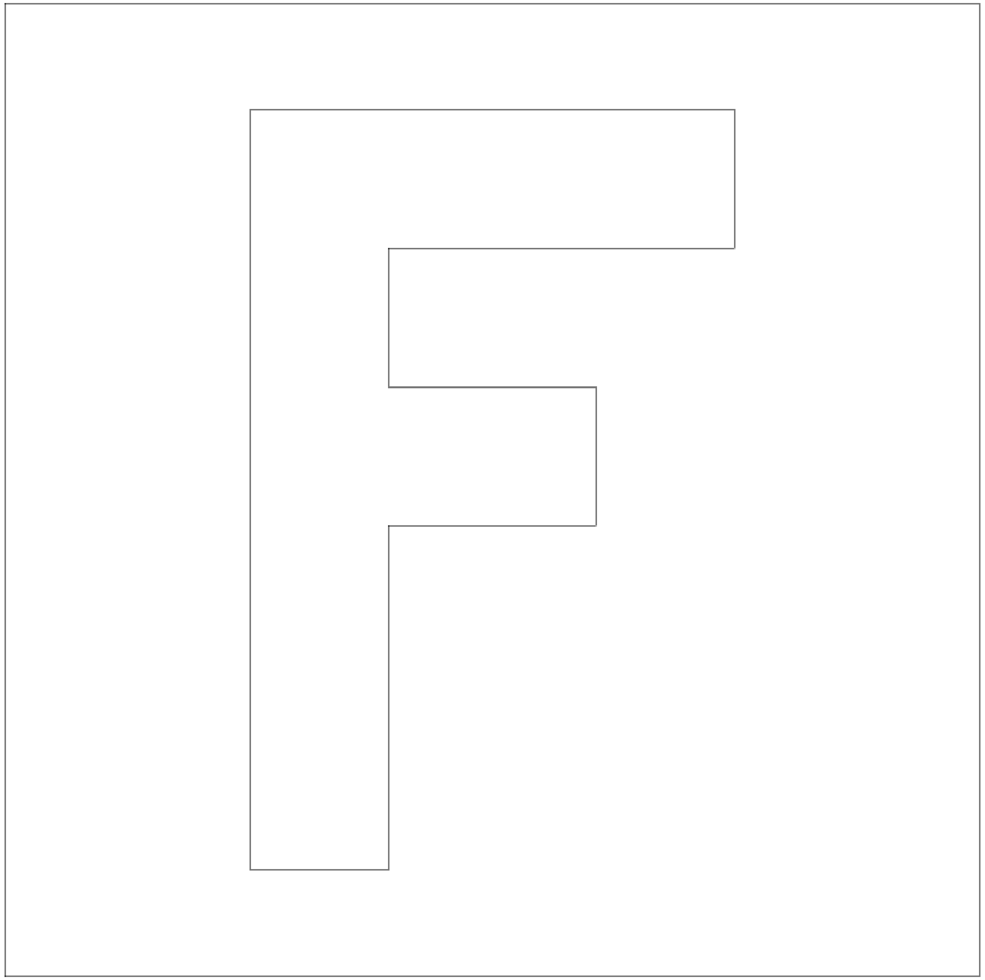
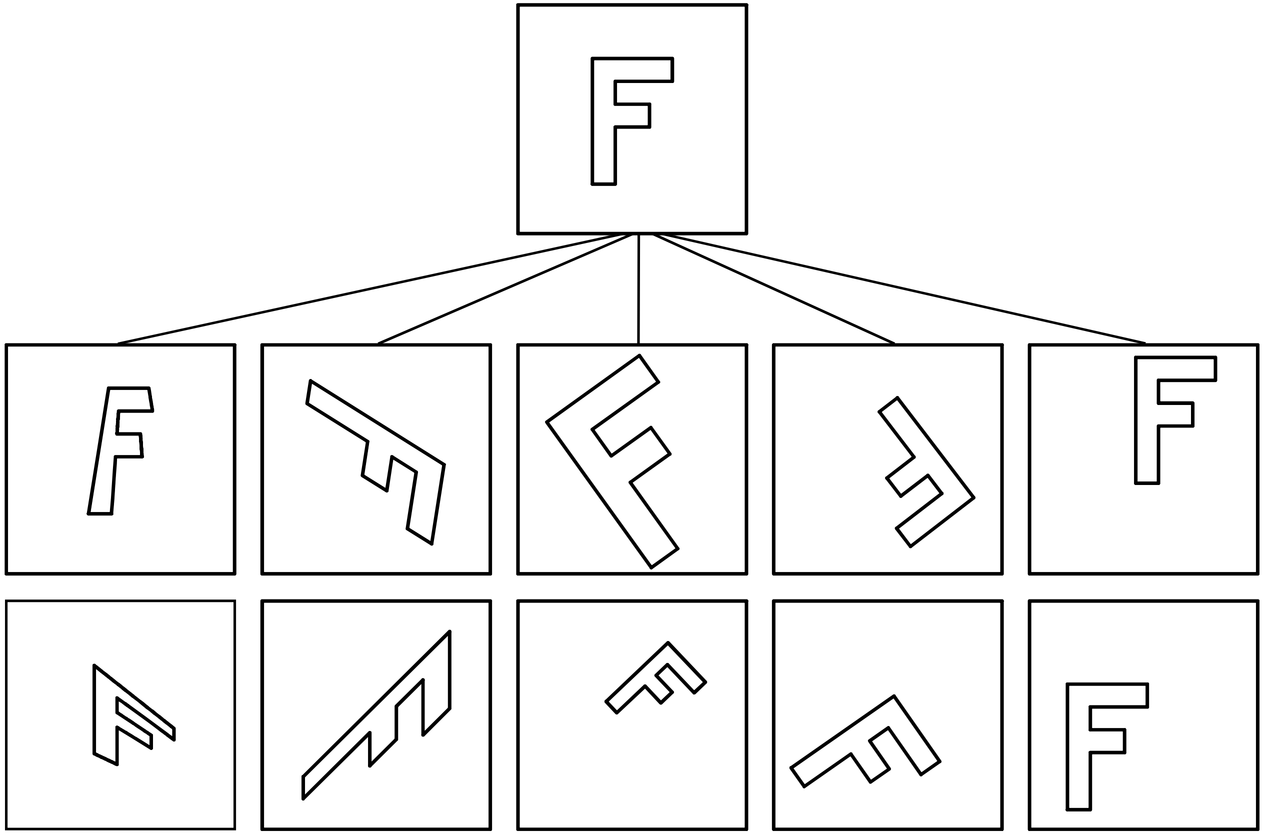
Affine invariant Fourier descriptors

Sought: a generalization of the previously introduced similarity-invariant Fourier descriptors



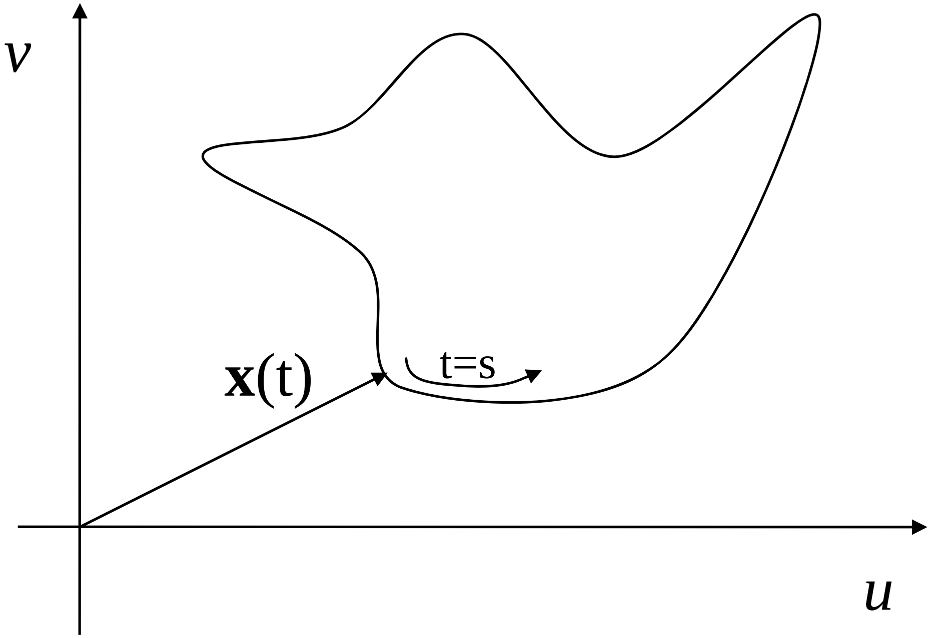
Geometrical transformations



central projections **affine mappings** **similarities** **congruencies** **translations**

(preserves parallelisms) (preserves angles)

Real, vectorial, parametric description of a closed contour



possible parameterization:
t=s (arc length)

Affine mapping of a contour

$$\boxed{\mathbf{x}(t) = \mathbf{A}\mathbf{x}^0(t(t^0)) + \mathbf{b}}$$

with: $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \det(\mathbf{A}) \neq 0$

Additonally starting point translation:

$$t(t^0, \tau)$$

In case arc length is used for

parameterization: $t(t^0, \tau) = t(t^0 + \tau)$

Thus 7 degrees of freedom result

for affine mapping!

Equivalent structures

- In the equivalence class of *similar* maps with equivalence relation ~ applies:

  circle1 ~ circle2
  circle $\nsim$ ellipse
  parallelogramm $\nsim$ rectangle $\nsim$ square

- In the equivalence class of *affine* maps though applies:

  circle ~ ellipse
  parallelogramm ~ rectangle ~ square
  but: circle $\nsim$ square

Developing the contour as a periodic function into a Fourier series

$$\mathbf{x}(t)=\begin{bmatrix}u(t)\\v(t)\end{bmatrix}=\sum_{k=-\infty}^{k=+\infty}\mathbf{X}_k e^{j2\pi kt/T}$$

with the complex valued Fourier coefficient vector:

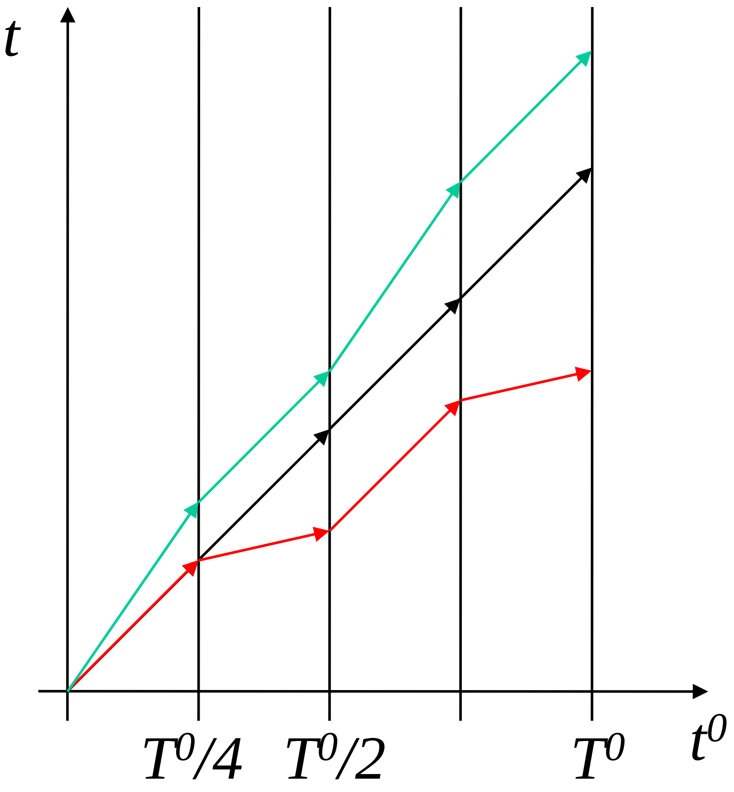
$$\mathbf{X}_k=\begin{bmatrix}U_k\\V_k\end{bmatrix}=\tfrac{1}{T}\int_{t=0}^{T}\mathbf{x}(t)e^{-j2\pi kt/T}dt$$

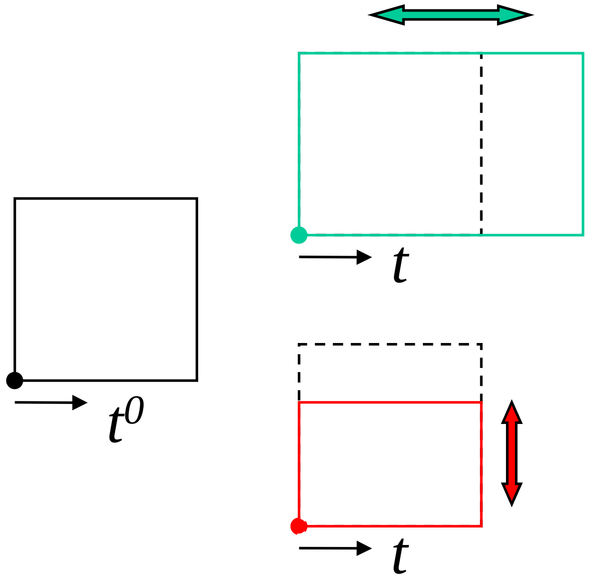
Choosing a parameterization, which guarantees a linear (homogeneous) mapping $t^0 \rightarrow t$ with the effect of the affine map **A**

$$t(t^0, \mathbf{A}) = \mu(\mathbf{A}) \cdot t^0$$

The arc length does not meet this requirement!

Non-linear map over arc length with shear of objects





horizontal
dilation

vertical
compression

Choosing an appropriate parameterization

1st possibility: Using differential invariants of second order in form of *affine length*. Needed are:

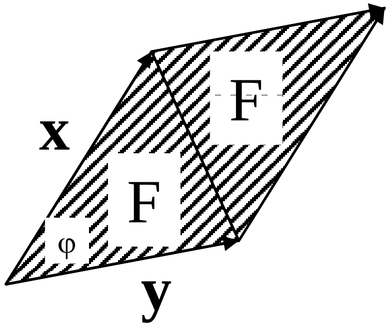
$$\left[\dot{\mathbf{x}}, \ddot{\mathbf{x}}\right]$$

2nd possibility: Using differential invariants of first order and additionally area centre of gravity $\mathbf{x}_s$ (semi-differential approach). Needed are:

$$\left[\mathbf{x}, \dot{\mathbf{x}}\right]$$

The outer (tensor) product and its geometric meaning

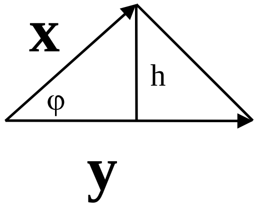
The outer product of two vectors [**x**,**y**] is a (signed) real number, which corresponds to the area of the included parallelogram (or: 2 times the area of the triangle)



outer product:

$$\left[\mathbf{x},\mathbf{y}\right] = \det(\mathbf{x},\mathbf{y}) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (x_1 y_2 - x_2 y_1) = \|\mathbf{x}\|\|\mathbf{y}\|\sin(\varphi)$$

$$\left\|\left[\mathbf{x},\mathbf{y}\right]\right\| = 2 \cdot F_\Delta$$



$$h = \|\mathbf{x}\|\sin(\varphi)$$

$$F_\Delta = \|\mathbf{y}\| \cdot h / 2 = \tfrac{1}{2}\|\mathbf{y}\|\|\mathbf{x}\|\sin(\varphi)$$

Results from differential geometry

We differentiate the parameterization *t* adequately from the arc length *s*.

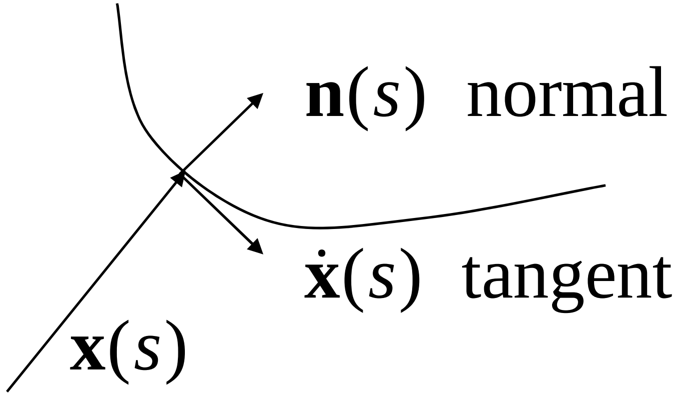
For an analytical curve (which is differentiable arbitrary number of times) applies by means of [**x**,**y**]:

$$dt(s) = \sqrt[(2n+1)]{\left[\mathbf{x}^{(n)},\mathbf{x}^{(n+1)}\right]}ds = \sqrt[(2n+1)]{\left[\mathbf{Ax}^{0^{(n)}},\mathbf{Ax}^{0^{(n+1)}}\right]}ds$$

$$= \underbrace{\sqrt[(2n+1)]{|\mathbf{A}|}}_{\mu(\mathbf{A})}\underbrace{\sqrt[(2n+1)]{\left[\mathbf{x}^{0^{(n)}},\mathbf{x}^{0^{(n+1)}}\right]}ds}_{dt^0} \qquad \text{n} \geq 1$$

with: $\mathbf{x}^{(n)} = \dfrac{d^n x}{ds^n}$ e.g. is $\mathbf{x}^{(1)} = \dot{\mathbf{x}}(s)$ tangential vector

$\mathbf{x}^{(2)} = \kappa(s)\mathbf{n}(s)$ curvature vector



$$\left\|\dot{\mathbf{x}}(s)\right\| = \left\|\mathbf{n}(\text{s})\right\| = 1$$

$$\Rightarrow \quad \boxed{dt = \mu(\mathbf{A})\cdot dt^0}$$

1st possibility: Using differential invariants of second order in form of affine length

$$t = \int_{\mathcal{C}} \sqrt[3]{\left[\dot{\mathbf{x}}, \ddot{\mathbf{x}}\right]} ds = \int_{\mathcal{C}} \sqrt[3]{\kappa(s)} ds \quad \text{affine length}$$

with: $\dot{\mathbf{x}} = \frac{d\mathbf{x}(s)}{ds}$ $(s \mathrel{\hat{=}}$ arc length$)$

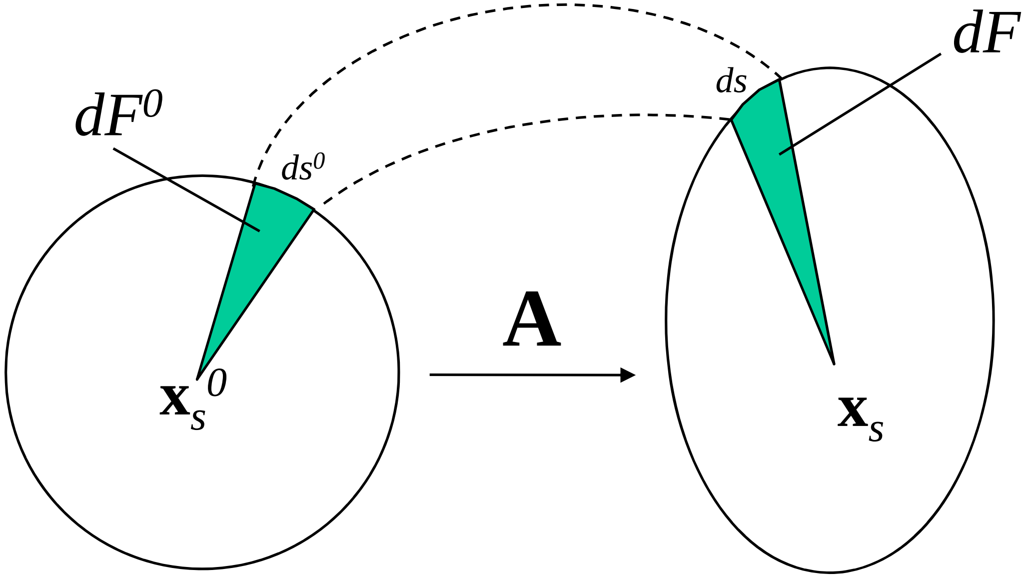
it applies: $$\underbrace{\int_{\mathcal{C}} \sqrt[3]{\left[\dot{\mathbf{x}}, \ddot{\mathbf{x}}\right]} ds}_{t} = \underbrace{\sqrt[3]{\left|\mathbf{A}\right|}}_{\mu(\mathbf{A})} \cdot \underbrace{\int_{\mathcal{C}} \sqrt[3]{\left[\dot{\mathbf{x}}^0, \ddot{\mathbf{x}}^0\right]} ds}_{t^0}$$

and thus: $$\boxed{t = \sqrt[3]{\left|\mathbf{A}\right|} t^0 = \mu(\mathbf{A}) t^0}$$

problem for polygons: the second derivative disappears along the line and the first derivative is non-continuous in the corners and therefore the 2nd derivative is not defined!

2nd possibility: using differential invariants of first order and additionally normalization by center of gravity (COG) $\mathbf{x}_s$ (semi-differential approach)

The vector starting from the COG to the contour are used for parameterization (outer product of pointer and tangential vector)



$$\boxed{\begin{aligned} dF &= \alpha(\mathbf{A}) \cdot dF^0 \\ &= \det(\mathbf{A}) \cdot dF^0 \end{aligned}}$$

$$t = F = \int_{\mathcal{C}} \overbrace{[\mathbf{x} - \mathbf{x}_s, \dot{\mathbf{x}}]}^{dF} ds$$

$$= |\mathbf{A}| \int_{\mathcal{C}} \overbrace{\left[\mathbf{x}^0 - \mathbf{x}_s^0, \dot{\mathbf{x}}_0\right]}^{dF^0} ds$$

$$= \mu(\mathbf{A}) \cdot F^0 = \mu(\mathbf{A}) \cdot t^0$$

due to:

$$\det(\mathbf{Ax}, \mathbf{Ay}) = \det(\mathbf{A} \cdot (\mathbf{x}, \mathbf{y})) = \det(\mathbf{A}) \cdot \det(\mathbf{x}, \mathbf{y})$$

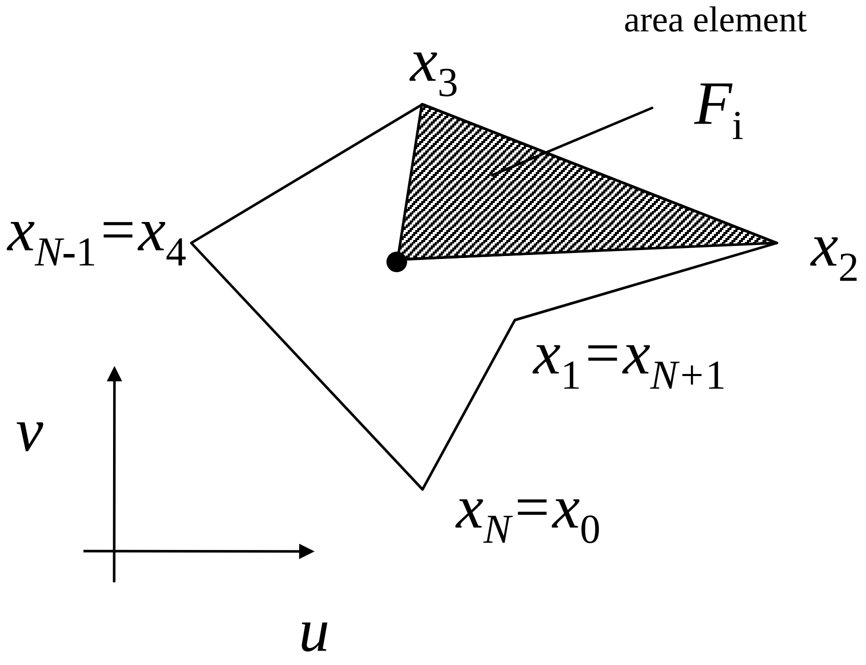
$$\Rightarrow \qquad [\mathbf{Ax}, \mathbf{Ay}] = \det(\mathbf{A}) \cdot [\mathbf{x}, \mathbf{y}]$$

The effect of the transformation is eliminated due to the normalization to the COG

It applies: The affine transformation maps area COG to each other and areas to each other in a constant relation!

The outer product is signed! In order to avoid ambiguities the amplitude of area increment $|dF|$ is chosen and therefore a monotonic increasing parameterization!

Affine invariant Fourier descriptors for polygons



polygon: $\left[\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_N\right]$ $\quad \mathbf{x}_i=\begin{bmatrix} u_i \\ v_i \end{bmatrix}$

Affine invariant Fourier descriptors of polygons

area center of gravity of the whole traverse:

$$\mathbf{x}_s = \tfrac{1}{3}\frac{\sum_{i=0}^{N-1}\overbrace{\left[\mathbf{x}_i,\mathbf{x}_{i+1}\right]}^{\det(\mathbf{x}_i,\mathbf{x}_{i+1})}(\mathbf{x}_i+\mathbf{x}_{i+1})}{\sum_{i=0}^{N-1}\left[\mathbf{x}_i,\mathbf{x}_{i+1}\right]} = \tfrac{1}{3}\frac{\sum_{i=0}^{N-1}(u_i v_{i+1}-u_{i+1}v_i)(\mathbf{x}_i+\mathbf{x}_{i+1})}{\sum_{i=0}^{N-1}\left[\mathbf{x}_i,\mathbf{x}_{i+1}\right]}$$

parameter: $t_0 = 0$

$$t_{i+1} = t_i + \underbrace{\tfrac{1}{2}\left|u_i' v_{i+1}' - u_{i+1}' v_i'\right|}_{F_i} \qquad i = 0,1,\ldots,N-1 \qquad \boxed{T = t_N}$$

$$\mathbf{x}' = \begin{bmatrix} u' \\ v' \end{bmatrix} = \mathbf{x} - \mathbf{x}_s = \begin{bmatrix} u - u_s \\ v - v_s \end{bmatrix}$$

Fourier coefficients

$$\mathbf{X}_0 = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \tfrac{1}{2T} \sum_{i=0}^{N-1} (\mathbf{x}_{i+1} + \mathbf{x}_i)(t_{i+1} - t_i)$$

$$\mathbf{X}_k = \begin{bmatrix} U_k \\ V_k \end{bmatrix} = \tfrac{T}{(2\pi k)^2} \sum_{i=0}^{N-1} \frac{(\mathbf{x}'_{i+1} - \mathbf{x}'_i)}{(t_{i+1} - t_i)} (e_{k,i+1} - e_{k,i})(1 - \delta(t_{i+1} - t_i))$$

$$+ \tfrac{j}{2\pi k} \sum_{i=0}^{N-1} (\mathbf{x}'_{i+1} - \mathbf{x}'_i) e_{k,i} \delta(t_{i+1} - t_i) \quad \text{for} \quad k \neq 0$$

with: $e_{k,i} = e^{-j2\pi k t_i / T}$

$$\delta(t_{i+1} - t_i) = \begin{cases} 1 & \text{if} \quad t_{i+1} = t_i \quad \text{(planar increase = 0)} \\ 0 & \text{if} \quad t_{i+1} \neq t_i \end{cases}$$

first part transforms continuities

second part transforms discontinuities

(switching through δ-operator)

Fourier coefficients of affine distorted contours

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}^0(t^0 + \tau) + \mathbf{b}$$

$$\mathbf{X}_k = \mathcal{F}(\mathbf{x}(t))$$

$$\mathbf{X}_k^0 = \mathcal{F}(\mathbf{x}^0(t^0))$$

thus follows:

$$\boxed{\begin{array}{l} \mathbf{X}_k = z^k \mathbf{A}\mathbf{X}_k^0 \qquad k \neq 0 \text{ (eliminates translation)} \\ z = e^{-j2\pi\tau/T} \end{array}}$$

A-invariants (τ=0)

with:

$$\Delta_{kp} = \det\left[\mathbf{X}_k, \mathbf{X}_p^*\right] = \det(\mathbf{A}) \cdot \det\left[\mathbf{X}_k^0, \mathbf{X}_p^{0\,*}\right] = \det(\mathbf{A}) \cdot \Delta_{kp}^0$$

from that result complete and minimal invariants:

$$Q_k = \frac{\Delta_{kp}}{\Delta_{pp}} = \frac{\det\left[\mathbf{X}_k, \mathbf{X}_p^*\right]}{\det\left[\mathbf{X}_p, \mathbf{X}_p^*\right]} = \frac{\cancel{\det(\mathbf{A})}}{\cancel{\det(\mathbf{A})}} \frac{\Delta_{kp}^0}{\Delta_{pp}^0} = \frac{U_k^0 V_p^{0*} - V_k^0 U_p^{0*}}{U_p^0 V_p^{0*} - V_p^0 U_p^{0*}} = Q_k^0$$

$p = const \neq 0 \qquad\qquad k = \pm 1, \pm 2, \pm 3, \ldots$

for $\tau \neq 0$ results though:

$$\boxed{Q_k = Q_k^0 \cdot z^{k-p} = Q_{kp}^0 \cdot z^k}$$

has to be eliminated

Additional starting point invariance ($\tau \neq 0$)

(special solution of second order)

$$\boxed{I_k = Q_k \Phi_q^{(k-p)\lambda} \Phi_r^{(k-p)\eta}}$$

with:

$$Q_k = \left|Q_k\right| \Phi_k = \left|Q_k\right| e^{j\arg(Q_k)}$$

(λ, η) are integral solutions of

the following linear diophantic equation:

$$\lambda(q-p) + \eta(r-p) + 1 = 0$$

a solution exists for:

$$\mathrm{gcd}(q-p, r-p) = 1$$

(solution with extended Euclidean algorithm)

These invariants are also complete and minimal!
The approach realizes also a compensation of phases, which are unknown mod 2π.

for example:

$$r = 7, q = 6, p = 1$$

$$\left.\begin{array}{l} q - p = 5 \\ r - p = 6 \end{array}\right\} \mathrm{gcd}(5,6) = 1$$

$$\Rightarrow \boxed{\lambda \cdot 5 + \eta \cdot 6 + 1 = 0}$$

holds for: $\lambda = 1, \eta = -1$

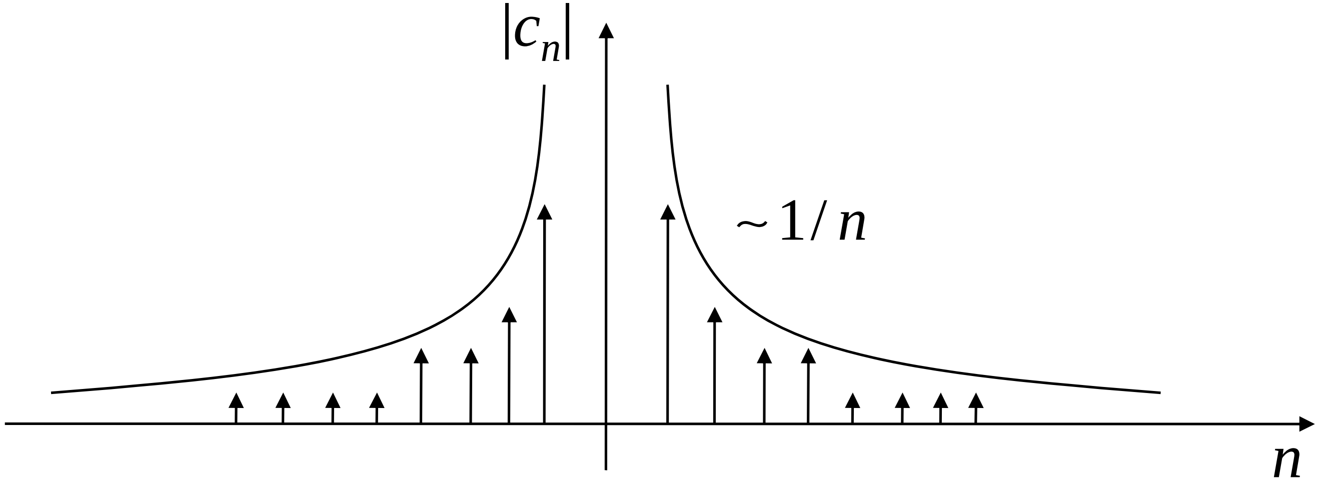
$$\Rightarrow \boxed{I_k = Q_k \Phi_6^{k-1} \Phi_7^{1-k}}$$

Also in this case one representative from the equivalence class results from the invariants, i.e. a contour in a certain location and view!
Also a linear complexity results for a constant number of Fourier descriptors :

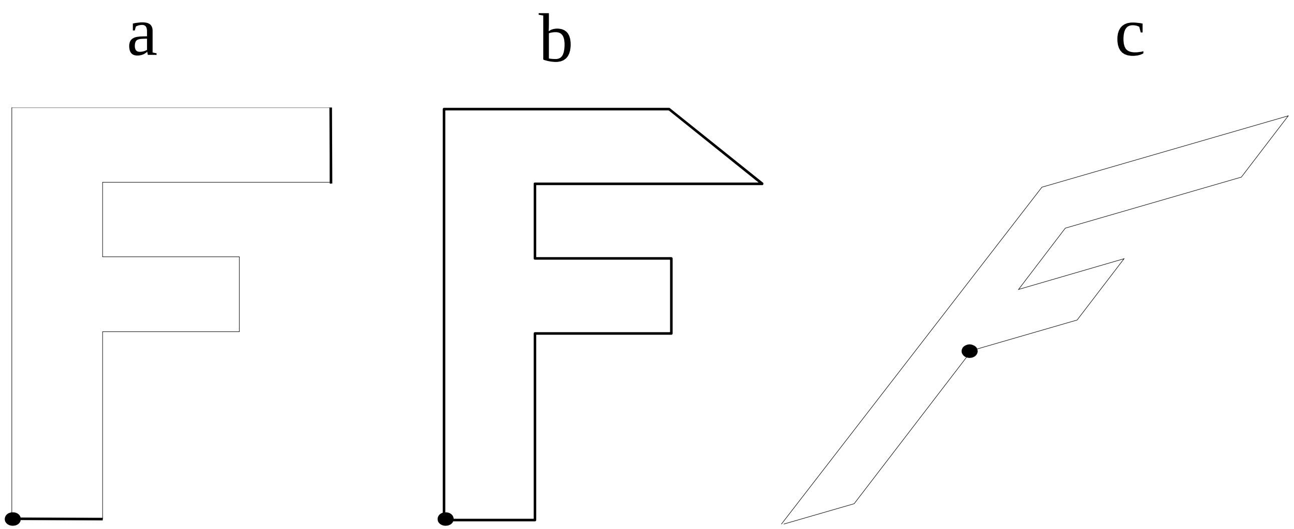
O(N)

Properties of Fourier series

Since the parametrical description of contours still contains discontinuities (polygon section in radial direction with planar increase 0), the magnitude of the FC is proportional to 1/*n* and thus tend to 0, which is slower than for continuous functions.



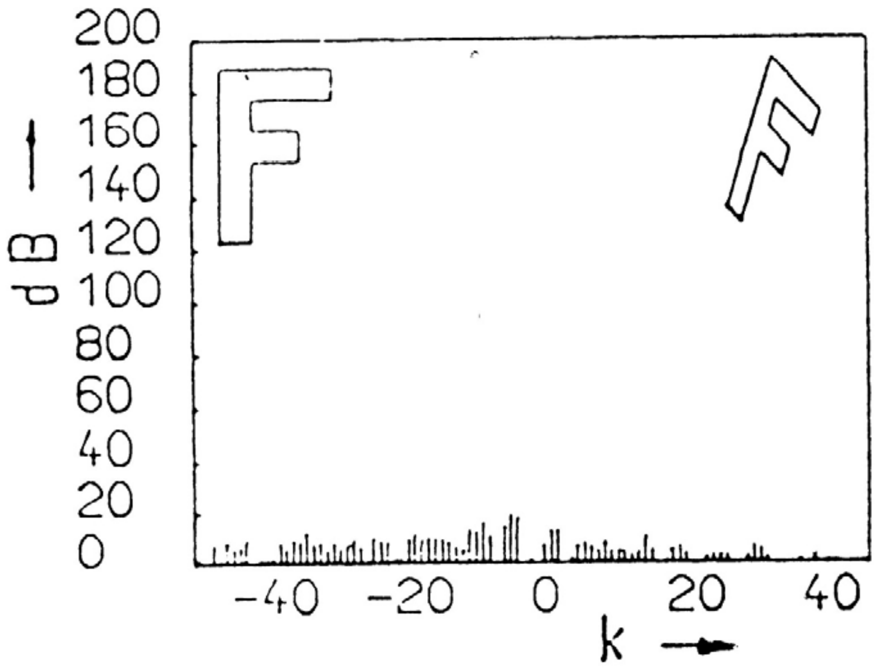
Affine invariant Fourier descriptors



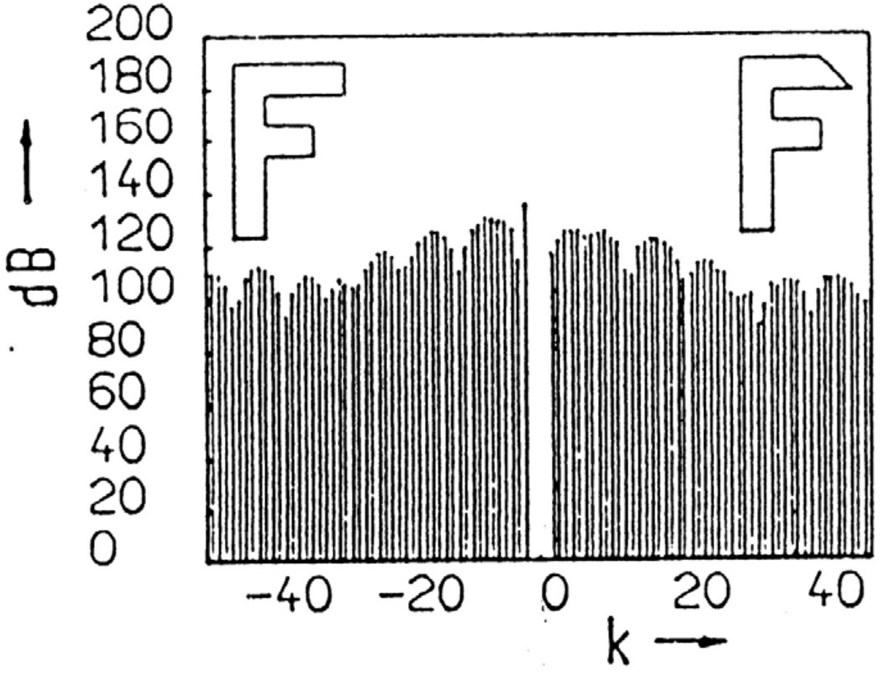
	Fourierkoeffizienten						Invarianten		
n	**a**		**b**		**c**		$\tilde{a}$	$\tilde{b}$	$\tilde{c}$
-5	0.107	0.004	0.146	-0.001	-0.086	0.284	**0.075**	0.094	**0.075**
-4	-0.006	-0.034	-0.047	-0.058	-0.086	0.311	**0.057**	0.084	**0.057**
-3	-0.036	-0.055	-0.001	-0.074	0.126	0.481	**0.029**	0.014	**0.029**
-2	0.283	-0.477	0.227	-0.490	-1.560	0.392	**0.315**	0.290	**0.315**
-1	-0.263	-0.779	-0.178	-0.733	-5.370	0.661	**0.000**	0.000	**0.000**
0	---	---	---	---	---	---	**---**	---	**---**
1	-1,120	-1,730	-1,090	-1,650	0,743	-7,330	**1,000**	1,000	**1,000**
2	-0.024	-0.375	-0.064	-0.467	0.927	-0.751	**0.229**	0.252	**0.229**
3	-0.169	-0.104	-0.191	-0.096	0.702	0.030	**0.104**	0.111	**0.104**
4	-0.081	0.182	-0.063	0.175	0.476	-0.385	**0.119**	0.126	**0.119**
5	0.066	-0.020	0.057	0.014	0.046	-0.201	**0.061**	0.059	**0.061**

$$\mathbf{F}' = \mathbf{A} \cdot \mathbf{F} \quad \text{mit: } \mathbf{F} = 0{,}5 \begin{bmatrix} 0 & 2 & 2 & 5 & 5 & 2 & 2 & 7 & 7(5) & 0 \\ 0 & 0 & 5 & 5 & 7 & 7 & 9 & 9 & 11 & 11 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$$

Power spectra of the difference of the invariants of both obiects



difference for real affine map considering the quantization error



difference for real structure changes