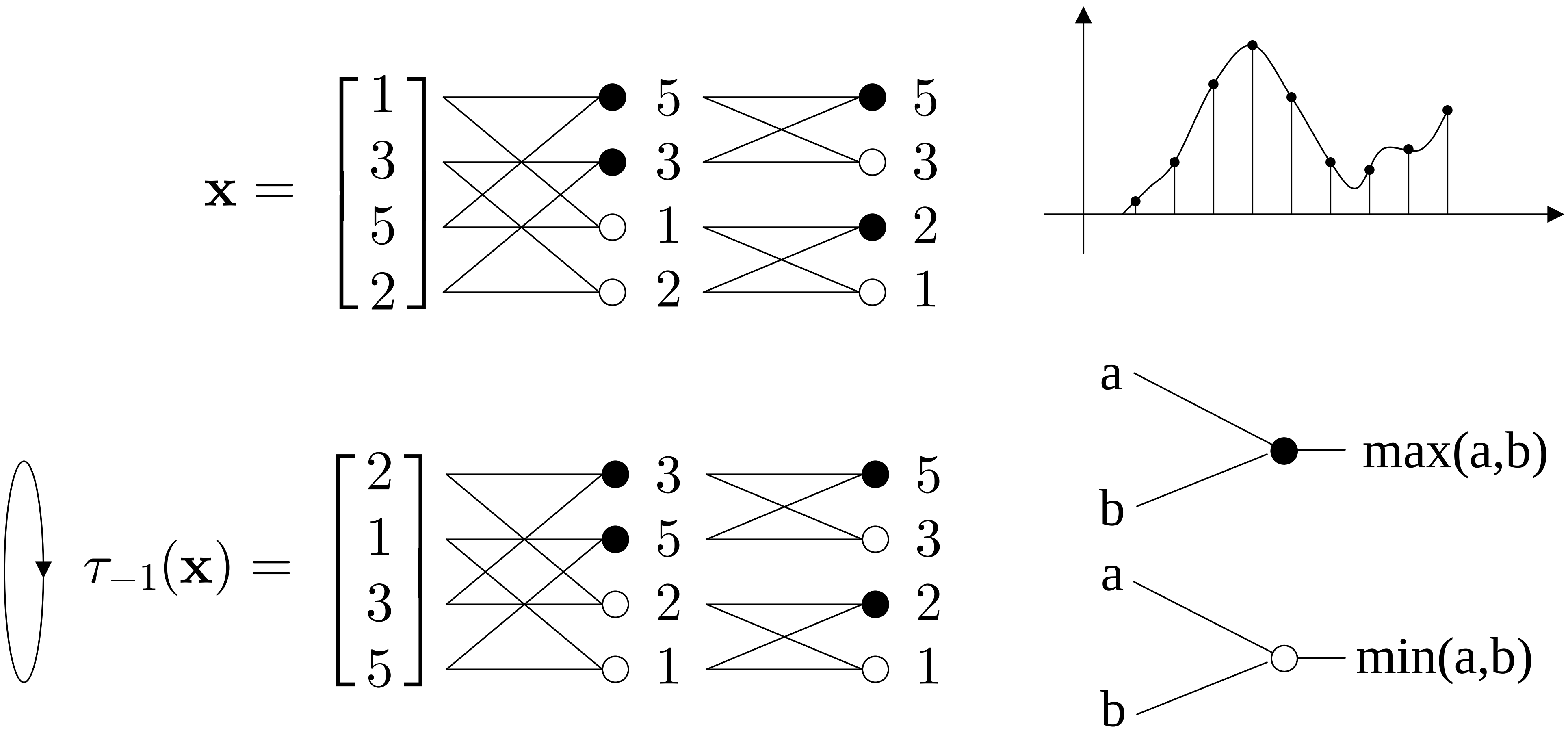
Chapter 3

Location-Invariant
Greyscale Recognition

One-dimensional translation-invariant features



Translation invariance of the Fourier transform‘s power spectrum and the autocorrelation function

The amplitude of the discrete Fourier transform (DFT, see DBV I) or also the power spectrum (amplitude‘s square) is translation invariant:

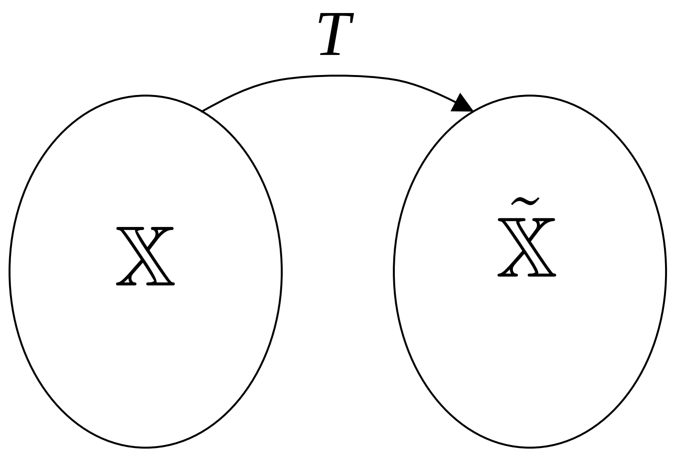
$$\text{AKF:}\quad \mathbf{z} = \mathbf{x} \# \mathbf{x} = \mathcal{F}^{-1}(\mathcal{F}(\mathbf{x}) \circ \mathcal{F}^{*}(\mathbf{x})) = \mathcal{F}^{-1}(\tilde{\mathbf{x}} \circ \tilde{\mathbf{x}}^{*}) = \mathcal{F}^{-1}(|\tilde{\mathbf{x}}|^{2})$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \end{bmatrix} \quad \mathbf{y} = \tau_{-1}(\mathbf{x}) = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \end{bmatrix} \quad \tilde{\mathbf{x}} = \begin{bmatrix} 11 \\ -4+j \\ 1 \\ -4-j \end{bmatrix} \quad \tilde{\mathbf{y}} = \begin{bmatrix} 11 \\ -1-4j \\ -1 \\ -1+4j \end{bmatrix}$$

$$|\tilde{\mathbf{x}}| = |\tilde{\mathbf{y}}| = \begin{bmatrix} 11 \\ 4,12 \\ 1 \\ 4,12 \end{bmatrix}$$

Cost of a general map from vector space $\mathbb{R}^N \rightarrow \mathbb{R}^N$ (linear or non-linear)

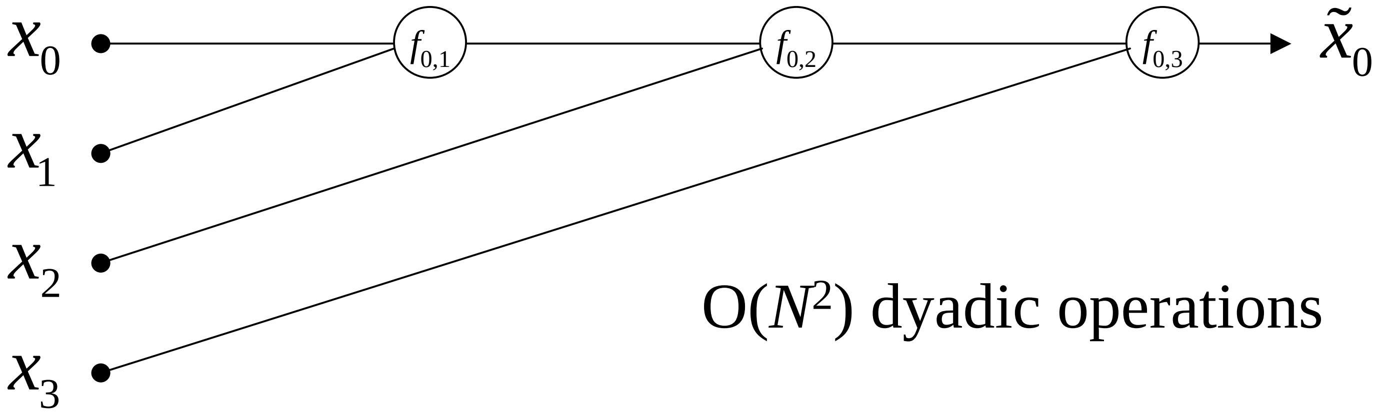
$$\tilde{\mathbf{x}} = T(\mathbf{x}) \quad \text{mit:} \quad \dim(\mathbf{x}) = \dim(\tilde{\mathbf{x}}) = N$$



The mapping T needs N^2 two-valued (dyadic) operations, if generally every input value affects every calculation of an output value.

The following schema shows the cost:

$$\tilde{x}_j = f_{j,N-1}(\cdots f_{j,3}(f_{j,2}(f_{j,1}(x_0, x_1), x_2), x_3)\cdots, x_{N-1})$$



O(N^2) dyadic operations

E.g. the linear vector space operation:

$$\boxed{\tilde{\mathbf{x}} = \mathbf{W}\mathbf{x}}$$

$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} \\ w_{10} & w_{11} & w_{12} & w_{13} \\ w_{20} & w_{21} & w_{22} & w_{23} \\ w_{30} & w_{31} & w_{32} & w_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\tilde{x}_0 = \overbrace{\underbrace{(\underbrace{(w_{00} \cdot x_0 + w_{01} \cdot x_1)}_{f_{01}(x_0, x_1)} + w_{02} \cdot x_2)}_{f_{02}(\bullet, x_2) = (\bullet) + w_{02} \cdot x_2} + w_{03} \cdot x_3}^{f_{03}(\bullet, x_3)}$$

Addition + multiplication counts as one operation.

A class of fast, non-linear, translation invariant transformation $\mathbb{C}T$

using recursive factorization of the transformation

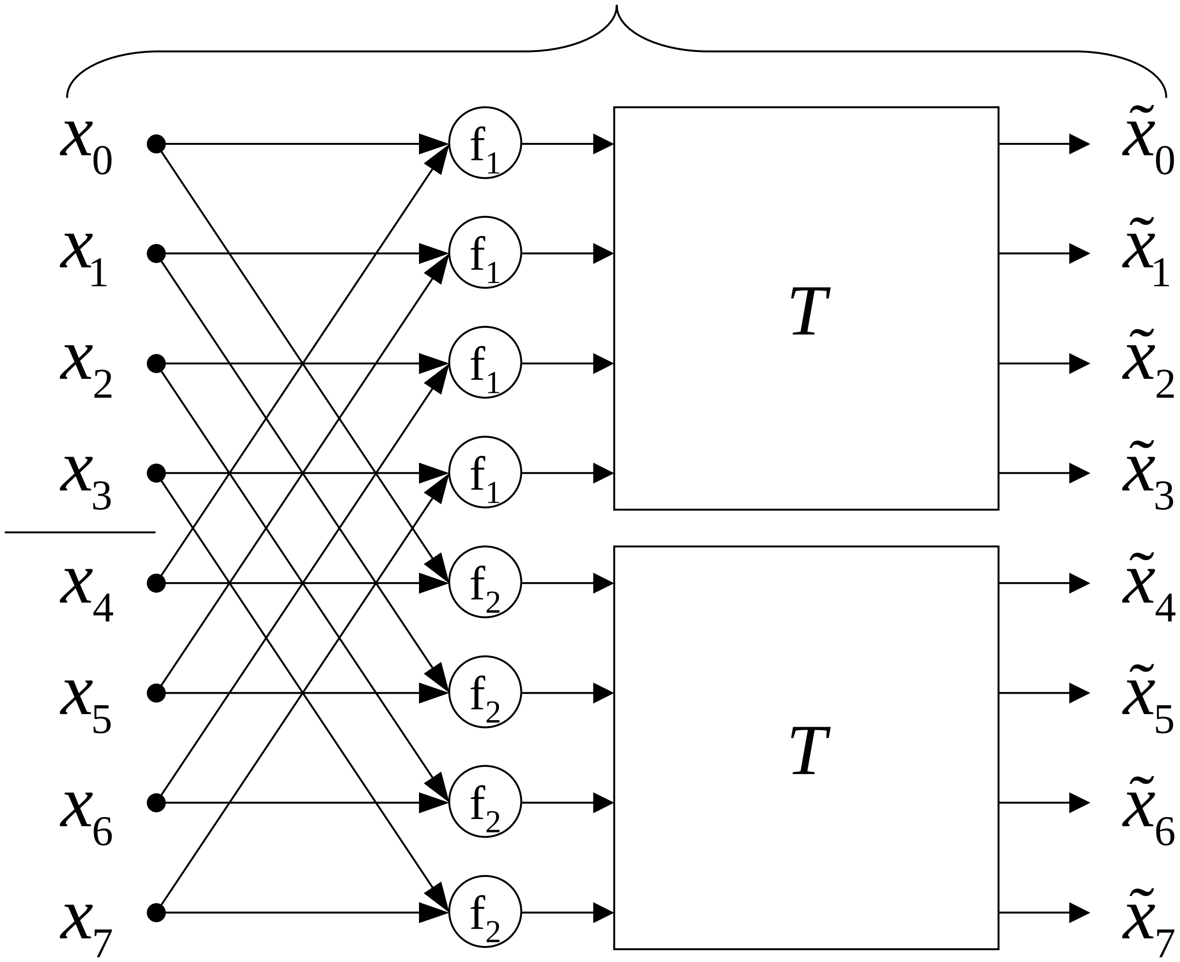
If the transformation can be factorized, i.e. the transformation of dimension *N* can be converted into two transformation of dimension *N/2* and then merged with linear cost, the result is:

$$\tilde{\mathbf{x}} = T(\mathbf{x}) = \begin{bmatrix} \widetilde{f_1(\mathbf{x}_{1|2}, \mathbf{x}_{2|2})} \\ \widetilde{f_2(\mathbf{x}_{1|2}, \mathbf{x}_{2|2})} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{x}_{1|2}^{(1)}} \\ \widetilde{\mathbf{x}_{2|2}^{(1)}} \end{bmatrix}$$

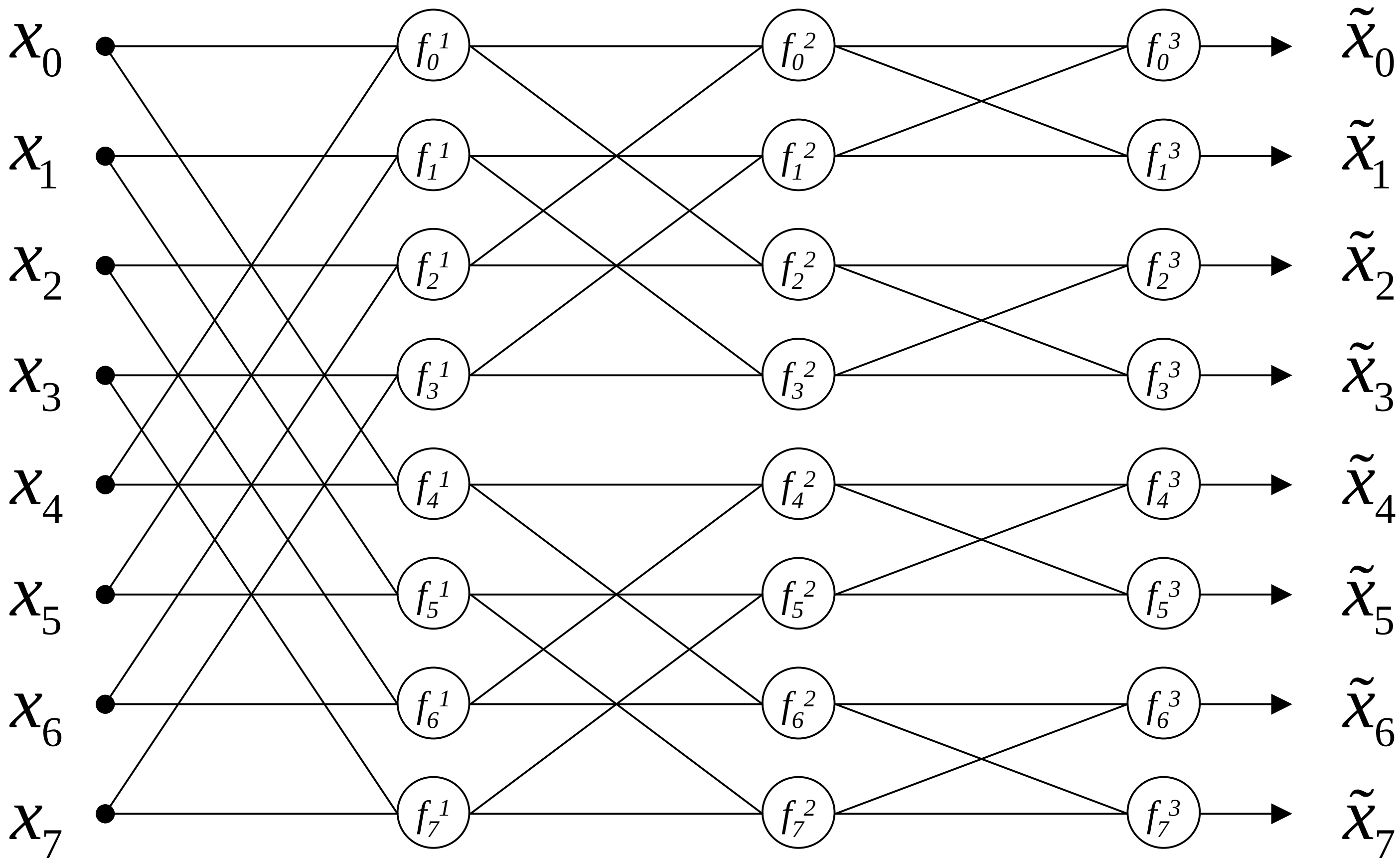
$f(\mathbf{x},\mathbf{y})$ denotes applying the two-valued operation f to the corresponding elements of the vectors **x** and **y**.

Recursive factorization of the transformation T

$$\tilde{\mathbf{x}} = T(\mathbf{x}) = \begin{bmatrix} \widetilde{f_1(\mathbf{x}_{1|2}, \mathbf{x}_{2|2})} \\ \widetilde{f_2(\mathbf{x}_{1|2}, \mathbf{x}_{2|2})} \end{bmatrix}$$



Resolving the recursion: Butterfly- or In-Place-information flow graph of the fast transformation T



Complexity

For $N=2^n$, factorization results in a cost of:

$$(1\cdot N+2\cdot N/2+4\cdot N/4+\cdots N/2\cdot 2)=N\cdot ld(N)=N\cdot n \text{ operations}$$

i.e. a cost improvement of: $$\frac{N^2}{N\log_2 N}=\frac{N}{\log_2 N}$$

For $N=2^{10}=1024$, a profit of $1024/10\approx 100$ fold.

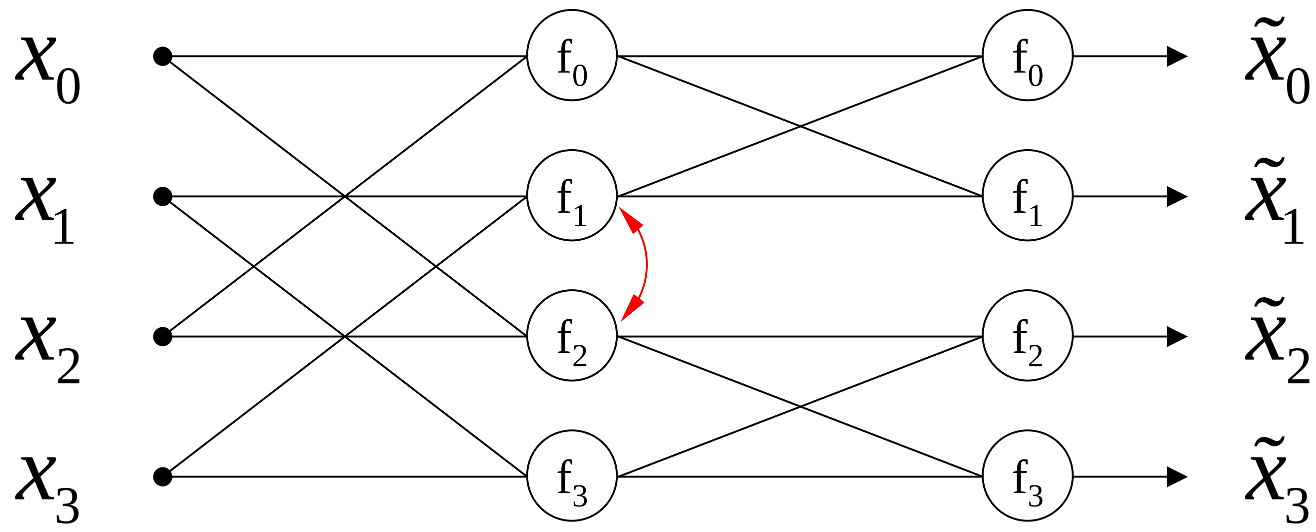
Running time improvement

N	N^2	$N \text{ ld } N$	profit: $\frac{N^2}{N \cdot \text{ld } N} = \frac{N}{\text{ld } N}$
100	10.000	664	15
500	250.000	4.483	55
1.000	10^6	10^4	100
$10^3 \cdot 10^3 = 10^6$	10^{12}	$20 \cdot 10^6$	50.000

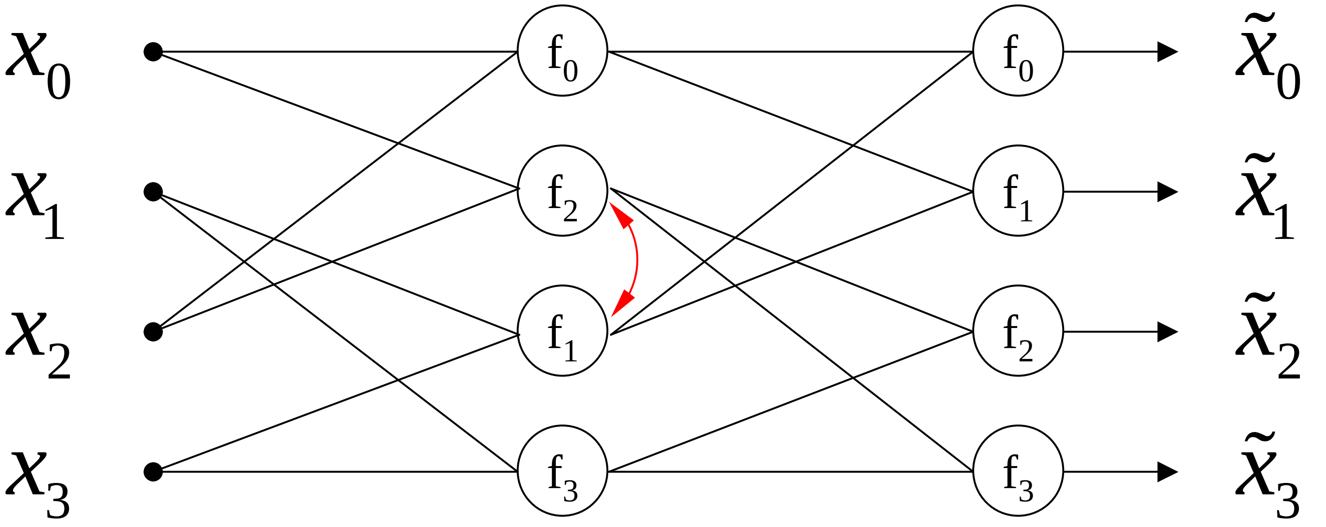
Consequences of factorization

1. Fast algorithm with $N\log_2 N$ two-valued operations
2. In-Place-algorithm
3. Modular use
   - Hardware: modular structure with components of smaller dimension
   - Software: modular use of smaller partial transformations e.g. with limited memory
4. Recursion very efficient for proof (complete induction)

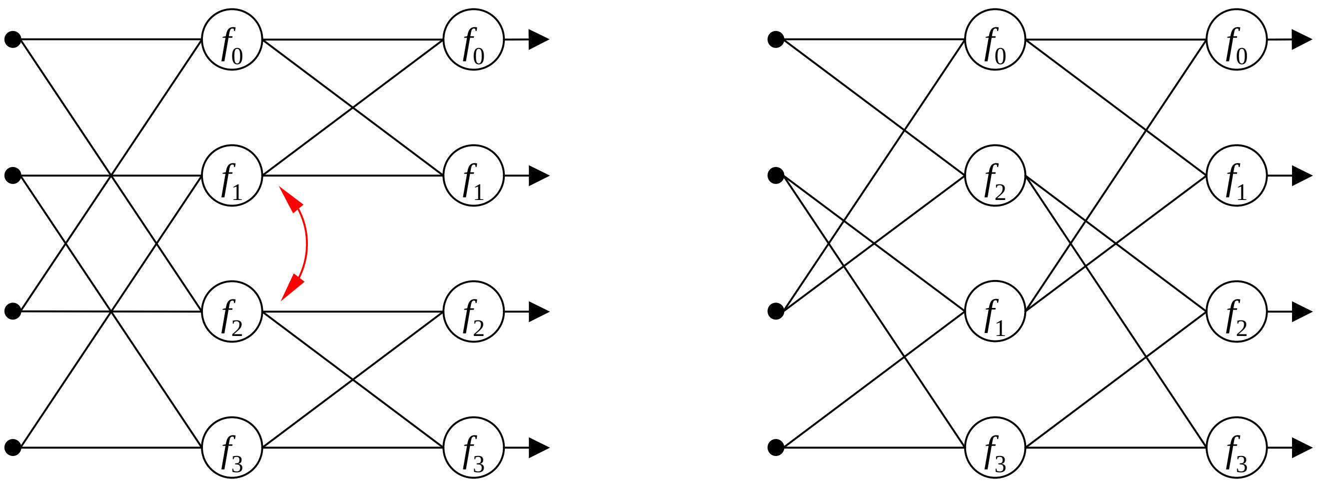
Generating a homogenous graph by permutation of nodes



Generating a homogenous graph by permutation of nodes



Generating a homogenous graph from a De Bruijn graph



Traverse nodes!

links are taken along

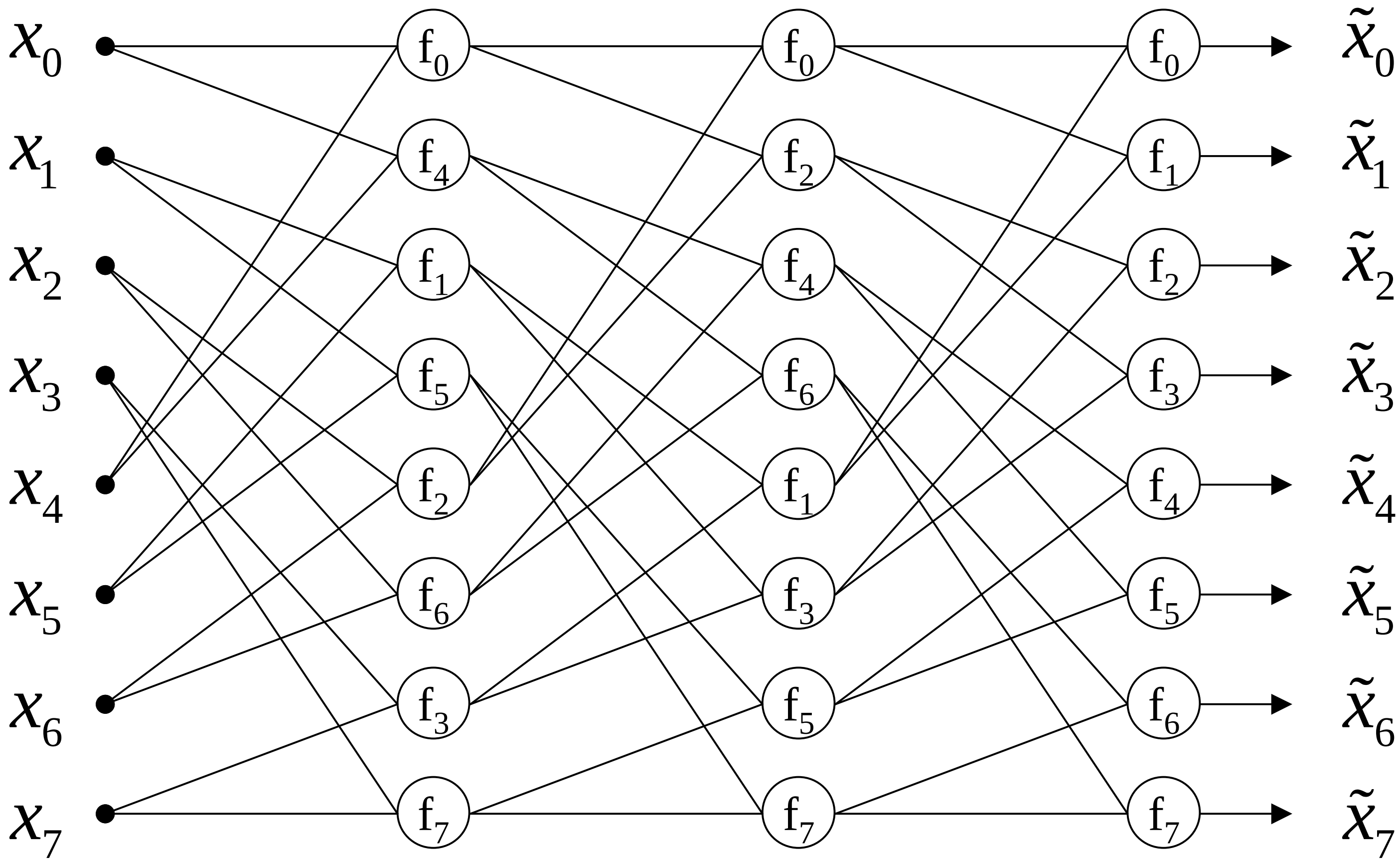
Changing the Butterfly- to the homogenous graph for a general base-B factorization by permutation of nodes

The operation f_r of layer j of the Butterfly-graph moves to r', with:

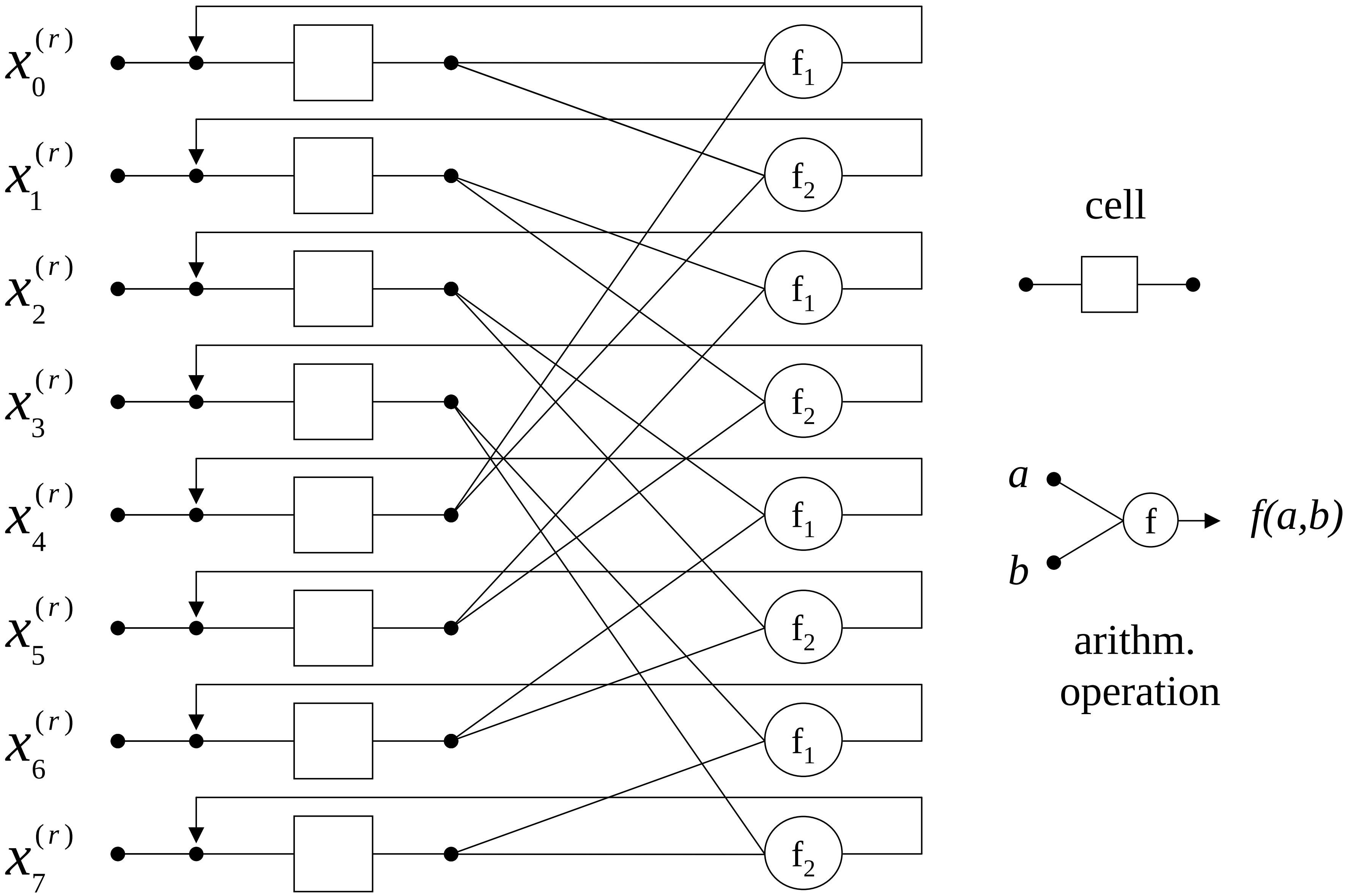
$$r'^{(j)} = \tau_j(r_B^{(0)}) = \tau_1(r_B^{(j-1)})$$

τ_j denotes j cyclic left shiftings starting at r, represented as B-adic numbers with $n=\log_B N$ digits

Homogenous or de Bruijn-information flow graph of the fast transform T



General parallel signal processor (N=8)



The transformation T in APL

```
[0] Z←T X;N
[1] A T RECURSIVE (with Bit reversal)
[2] →(1>N ←(,ρZ ← X)÷2)/0
[3] Z ←,(T((N ↑ X) F1 (N↓X))),[1.5](T((N ↑ X) F2 (N ↓ X)))
```

```
[0] Z ← T X;I;LN;NH
[1] A T ITERATIV
[2] LN ← 1+2μNH ←(ρ Z ← X) ÷ 2
[3] I ← 1
[4] M:Z ←,((NH ↑ Z) F1 (NH ↓ Z)),[1.5]((NH ↑ Z) F2 (NH ↓ Z))
[5] →(LN≥I ← I+1)/M
```

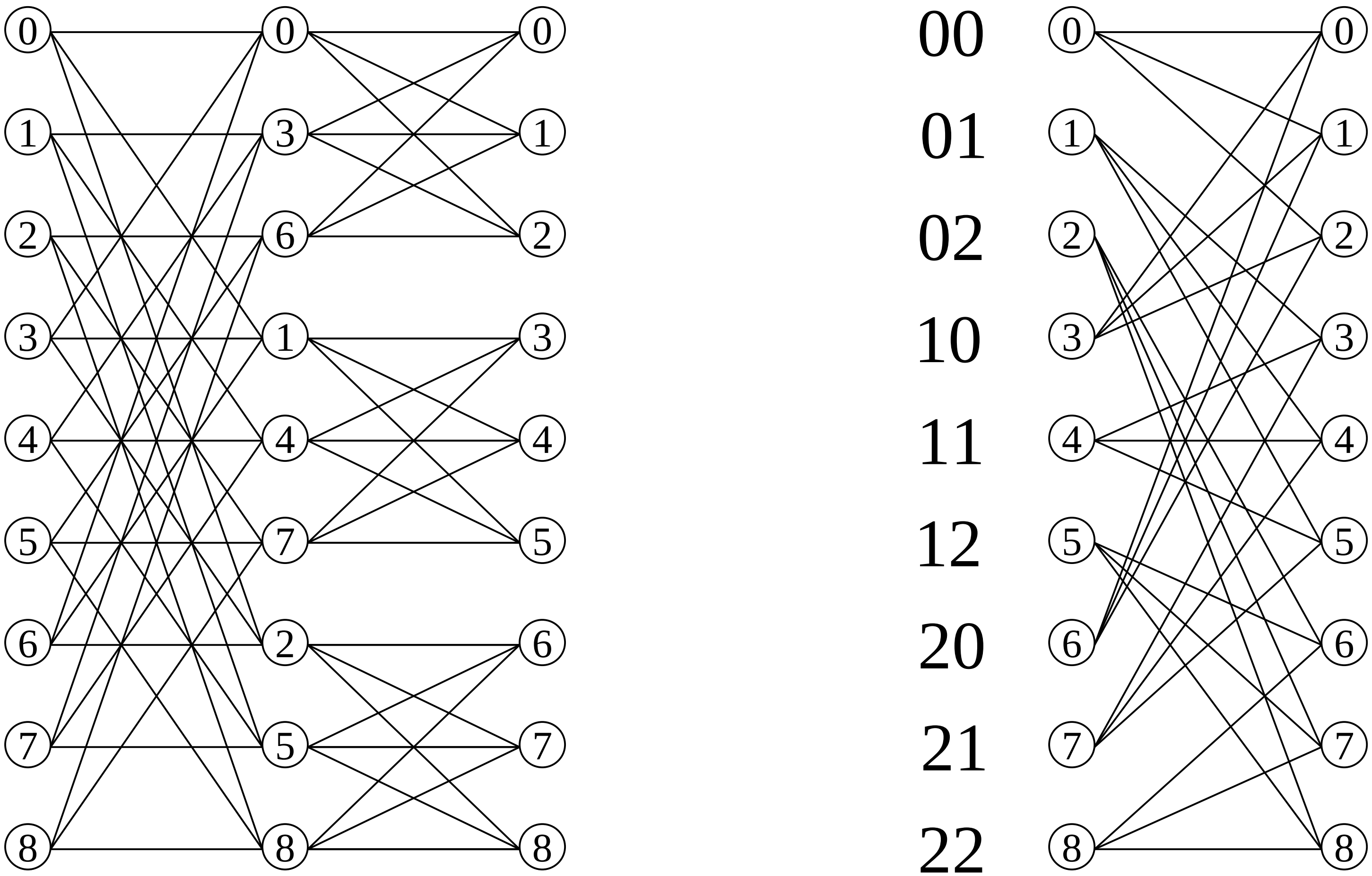
Recursive base-3 factorization of transformation *T*

$$\tilde{\mathbf{x}} = T(\mathbf{x}) = \begin{bmatrix} \overline{f_1(\mathbf{x}_{1|3}, \mathbf{x}_{2|3}, \mathbf{x}_{3|3})} \\ \overline{f_2(\mathbf{x}_{1|3}, \mathbf{x}_{2|3}, \mathbf{x}_{3|3})} \\ \overline{f_3(\mathbf{x}_{1|3}, \mathbf{x}_{2|3}, \mathbf{x}_{3|3})} \end{bmatrix}$$

$f(\mathbf{x},\mathbf{y},\mathbf{z})$ denotes applying the three-valued operation f to corresponding elements of the vectors **x**,**y** **a**nd **z**.

Exercise: Proof of translation invariance with symmetric function f(a,b,c), e.g. (a·b·c) and (a+b+c).

The two canonical graphs for a base-3 factorization



After at least ($\log_B N$) layers, every input element $\boldsymbol{x}_i$ affects every calculation of an output element $\tilde{x}_j$!

The class of fast, non-linear, translation invariant transformations $\mathbb{C}T$

The translation invariance results from requiring two-valued ***commutative*** operations for f_1 and f_2 :

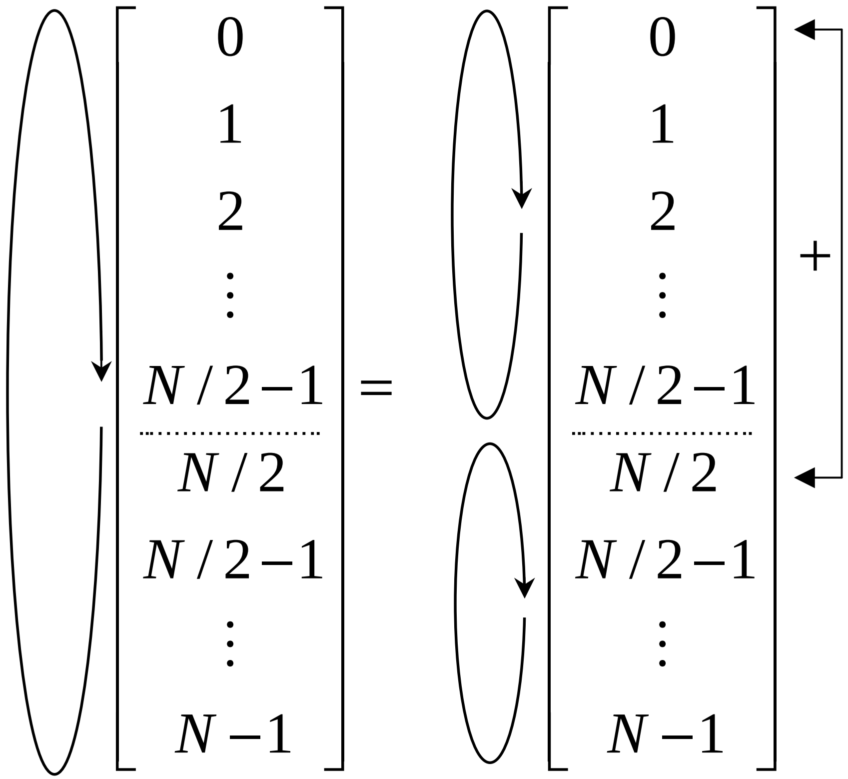
$$f_{1,2}(a,b) = f_{1,2}(b,a)$$

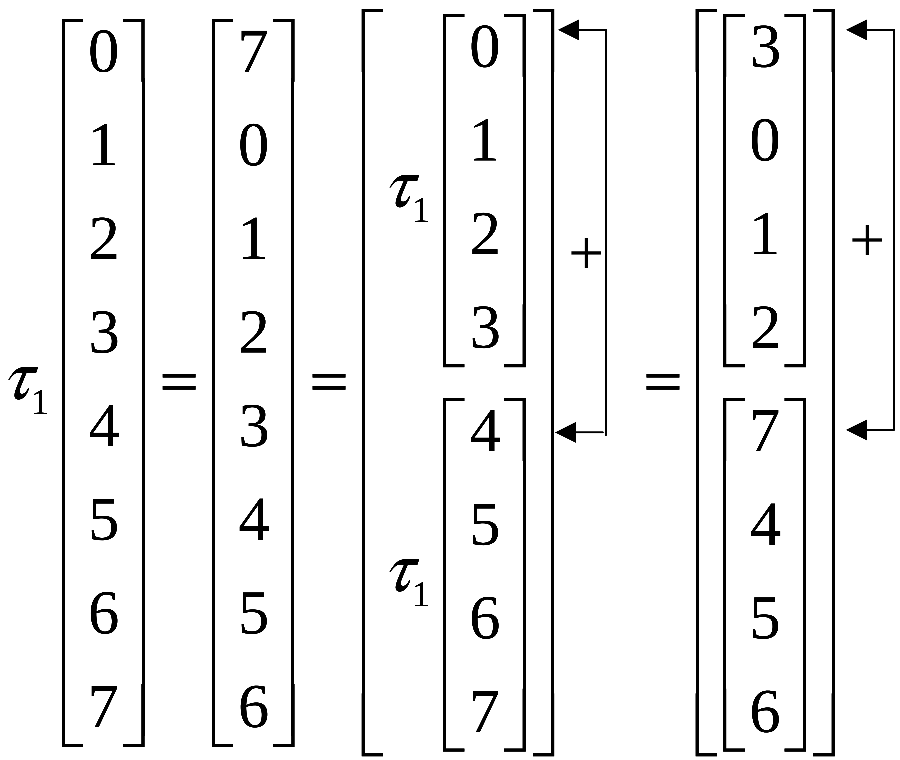
Examples:

	RT	QT	MT	BT
$f_1(a,b)$	$a+b$	$a+b$	$\max(a,b)$	$a \underset{n}{\wedge} b$
$f_2(a,b)$	$\lvert a-b \rvert$	$(a-b)^2$	$\min(a,b)$	$a \underset{n}{\vee} b$
definded for:	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{I}_2$

Proof of translation invariance for the class ℂT

Idea: a cyclic permutation of length N can be divided into two cyclic permutation of length N/2 mit subsequent permutation of the elements 0 and **N**/2:





Proof of translation invariance of the class ℂT

Sentence: For the class ℂT applies: $\widetilde{\tau_1(\mathbf{x})} = \tilde{\mathbf{x}}$

Proof with complete induction. The proposition results from the commutativity of the functions f_1 and f_2 for a fixed value N=2, i.e:

$$\widetilde{\tau_1(\mathbf{x})} = \widetilde{\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}} = \begin{bmatrix} \widetilde{f_1(x_1, x_0)} \\ \widetilde{f_2(x_1, x_0)} \end{bmatrix} = \begin{bmatrix} \widetilde{f_1(x_0, x_1)} \\ \widetilde{f_2(x_0, x_1)} \end{bmatrix} = \tilde{\mathbf{x}}$$

The inductive step is from $N/2$ to N ((n-1) →n). The base is:

$$\widetilde{\tau_1(\mathbf{x}_{1|2}^{(1)})} = \widetilde{\mathbf{x}_{1|2}^{(1)}}$$

$$\widetilde{\tau_1(\mathbf{x}_{2|2}^{(1)})} = \widetilde{\mathbf{x}_{2|2}^{(1)}}$$

For dimension N results:

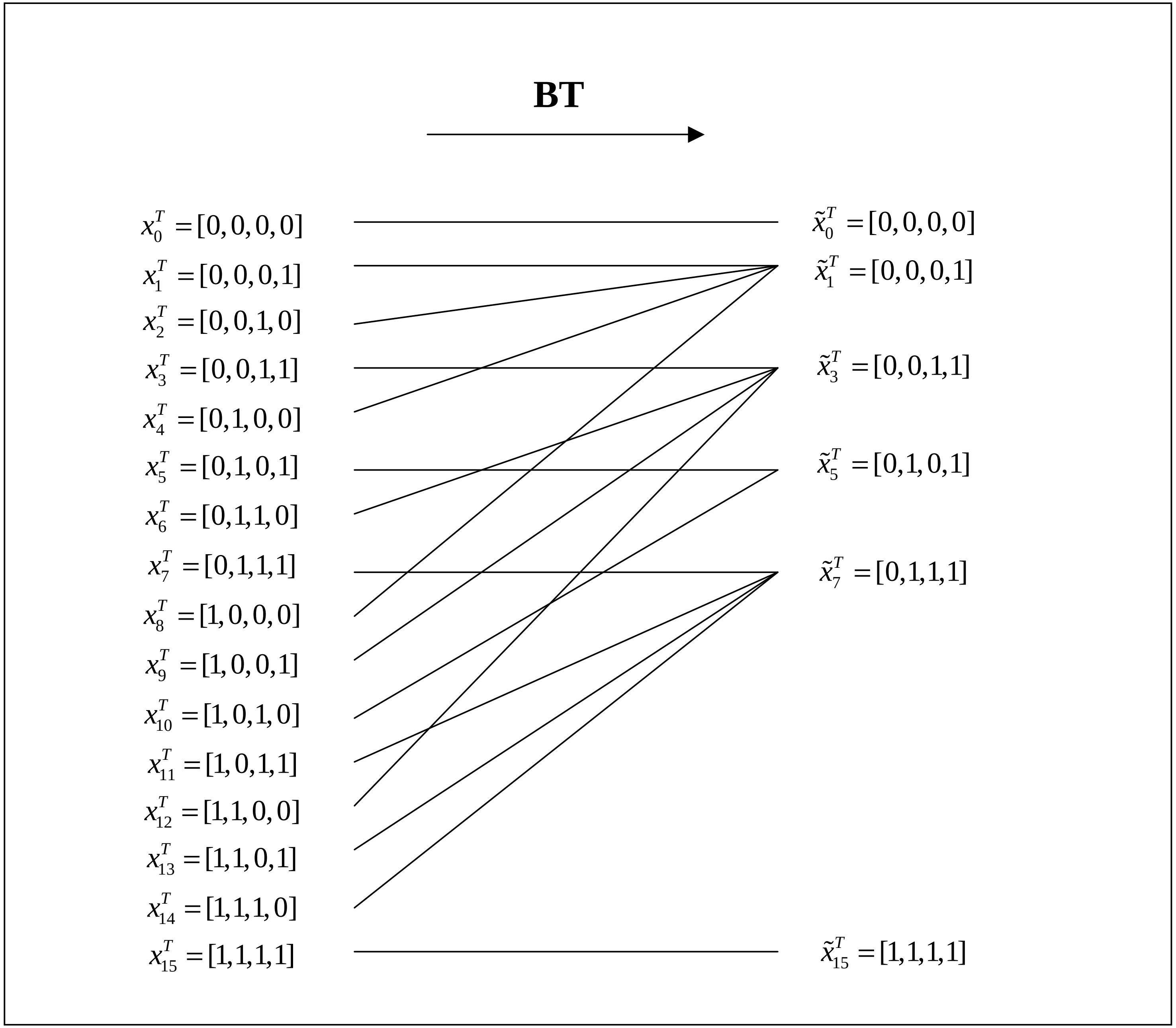
$$\widetilde{\tau_1(\mathbf{x})} = \begin{bmatrix} \widetilde{f_1\begin{bmatrix} x_{N-1}, x_{N/2-1} \\ x_0, x_{N/2} \\ x_1, x_{N/2+1} \\ \vdots \\ x_{N/2-2}, x_{N-2} \end{bmatrix}} \\ \widetilde{f_2\begin{bmatrix} x_{N-1}, x_{N/2-1} \\ x_0, x_{N/2} \\ x_1, x_{N/2+1} \\ \vdots \\ x_{N/2-2}, x_{N-2} \end{bmatrix}} \end{bmatrix} \overset{f_1, f_2 \text{ commutative}}{=} \begin{bmatrix} \widetilde{f_1\begin{bmatrix} x_{N/2-1}, x_{N-1} \\ x_0, x_{N/2} \\ x_1, x_{N/2+1} \\ \vdots \\ x_{N/2-2}, x_{N-2} \end{bmatrix}} \\ \widetilde{f_2\begin{bmatrix} x_{N/2-1}, x_{N-1} \\ x_0, x_{N/2} \\ x_1, x_{N/2+1} \\ \vdots \\ x_{N/2-2}, x_{N-2} \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} \widetilde{\tau_1(\mathbf{x}_{1|2}^{(1)})} \\ \widetilde{\tau_1(\mathbf{x}_{2|2}^{(1)})} \end{bmatrix} \overset{\text{base}}{=} \begin{bmatrix} \widetilde{\mathbf{x}_{1|2}^{(1)}} \\ \widetilde{\mathbf{x}_{2|2}^{(1)}} \end{bmatrix} = \tilde{\mathbf{x}}$$

qed.

This holds also for 2 or k translations:

$$\widetilde{\tau_2(\mathbf{x})} = \widetilde{\tau_1(\tau_1(\mathbf{x}))} = \widetilde{\tau_1(\mathbf{x})} = \tilde{\mathbf{x}}$$

and also: $\widetilde{\tau_k(\mathbf{x})} = \tilde{\mathbf{x}}$



B-transformation of all one-dimensional binary patterns, for N=4

(complete mapping for N=4!)

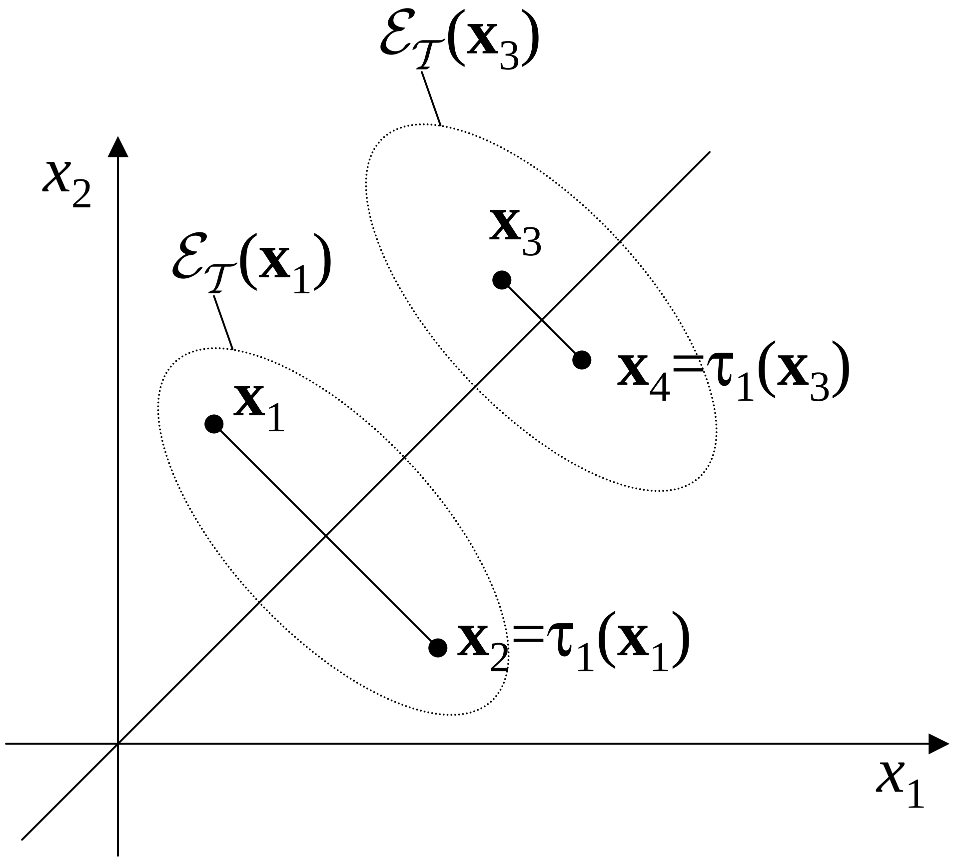
Discussing the properties of the R-transformation

For R-transformation holds:

$$f_1(a,b) = a + b$$
$$f_2(a,b) = |a - b|$$

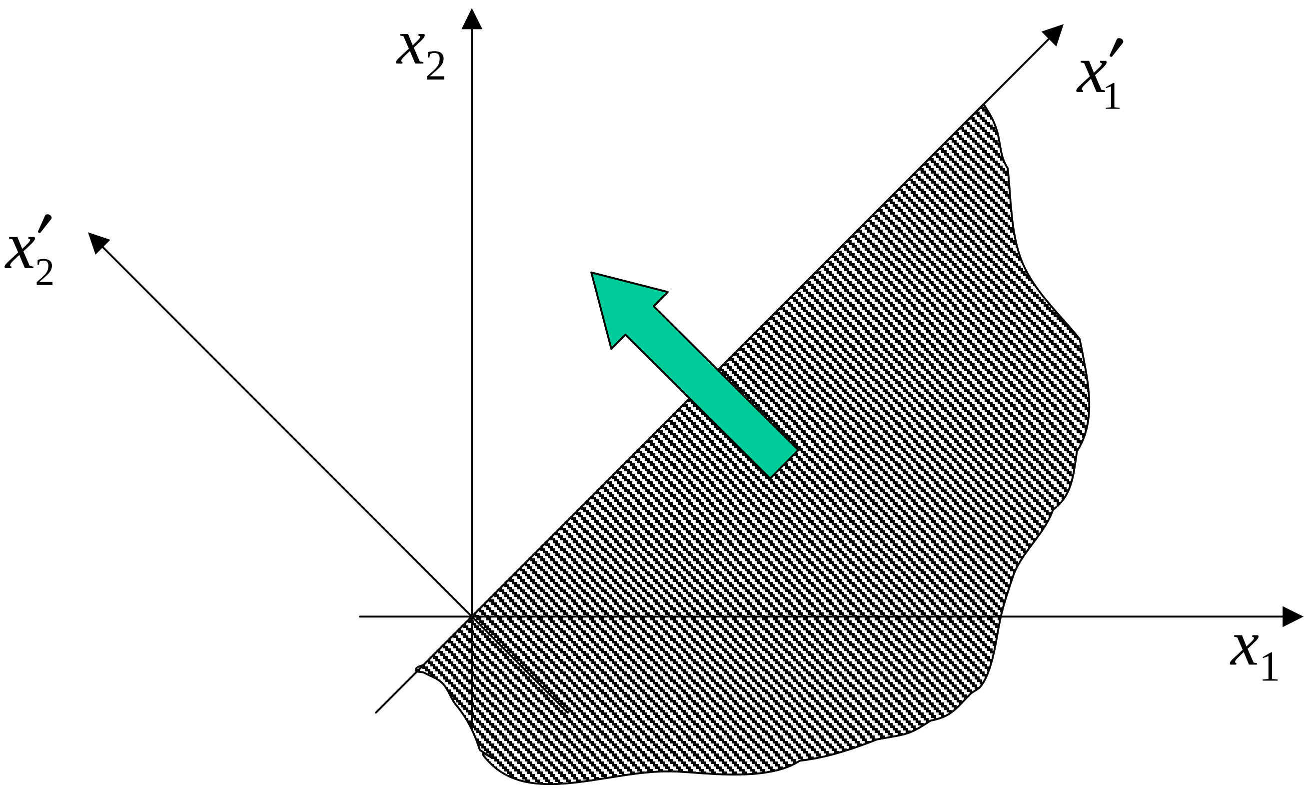
Geometric interpretation of the R-transformation for the two-dimensional Euclidian space $\mathbb{R}^2$

The effect of the R-Transformation on a two-dimensional Euclidian space can be geometrically interpreted in a simple way. The equivalence class can be retrieved by reflection on the angle bisection of the first quadrant.



Invariance through rotation of the coordinate system about 45° and convolution of the lower half of the plane and the upper

The effect of the R-Transformation on a two-dimensional Euclidian space can be geometrically interpreted in a simple way. The equivalence class can be retrieved by reflection on the angle bisection of the frist quadrant.



Rotation of the coordinate system about 45° and convolution of the lower half of the plane and the upper

Rotation about 45°:

$$x_1' = x_1 \cos(\pi/4) + x_2 \sin(\pi/4) = \frac{1}{2}\sqrt{2}(x_1 + x_2)$$

$$x_2' = -x_1 \sin(\pi/4) + x_2 \cos(\pi/4) = -\frac{1}{2}\sqrt{2}(x_1 - x_2)$$

Convolution of the upper half of the plane by $x_2' \to |x_2'|$. If the norm of the transformation is not obtained, the following results:

$$\tilde{x}_1 = x_1 + x_2$$

$$\tilde{x}_2 = |x_1 - x_2|$$

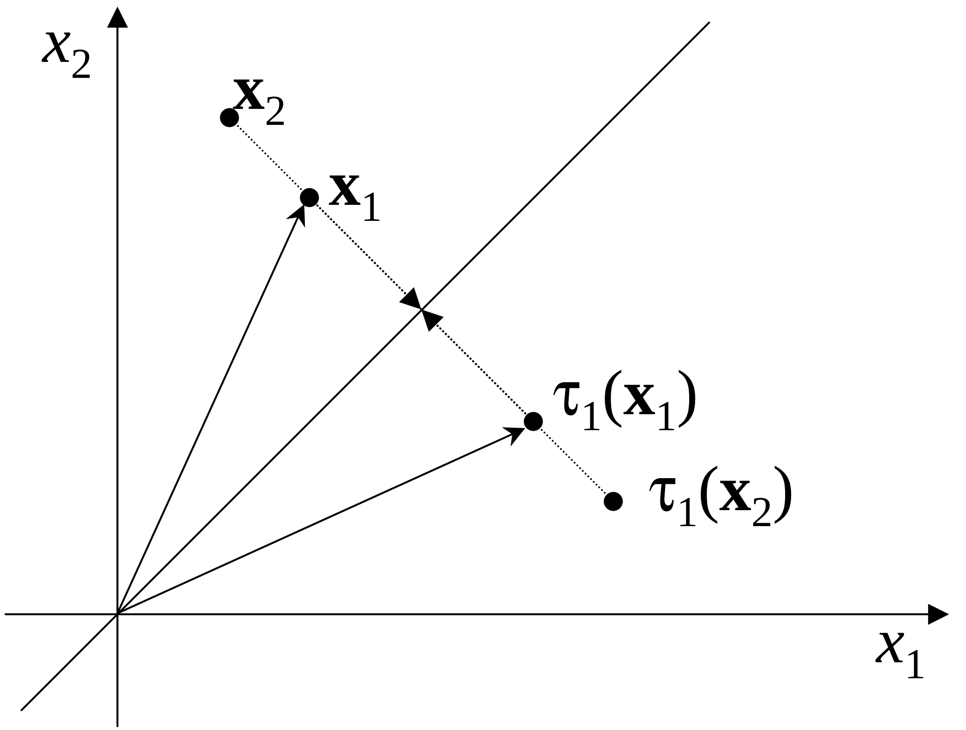
which is the definition of the R-transformation für N=2. Completeness is obvious:

$$\mathcal{E}_{\mathcal{T}}(\mathbf{x}_i) \Leftrightarrow \tilde{\mathbf{x}}_i$$

Convolution of the upper plane can be achieved by quadrature instead of building the absolute value. (Q-transformation): $\tilde{x}_2 = (x_1 - x_2)^2$

Example of an incomplete translation invariant transformation

The neccessary condition invariance can be retrieved by projection on the angle bisection of the first quadrant:



it applies: $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\varphi)$

$$\Rightarrow \tilde{x}_1 = \frac{1}{2}\sqrt{2}(x_1 + x_2)$$

The neccessary condition translation invariance is satisfied, but not completeness.

Completeness of RT

RT is in general not complete, e.g. for N=4:

$$\mathbf{x}_1 = \begin{bmatrix} 8 \\ 3 \\ 5 \\ 1 \end{bmatrix} \text{ and: } \mathbf{x}_2 = \begin{bmatrix} 3,5 \\ 7,5 \\ 0,5 \\ 5,5 \end{bmatrix} \text{ follows: } \tilde{\mathbf{x}}_1 = \tilde{\mathbf{x}}_2 = \begin{bmatrix} 17 \\ 9 \\ 5 \\ 1 \end{bmatrix}$$

Are there more vectors, that have the same transformed features?

The map properties for the class RT of dimension N=4 can be described as follows:

(Without proof) applies: $$\mathbb{I}_{RT}(\mathbf{x}_1) = \mathbb{I}_{\mathbb{C}T}(\mathbf{x}_1) \cup \mathbb{I}_{\mathbb{C}T}(\mathbf{x}_2)$$

with:

$$\mathbb{I}_{\mathbb{C}T}({}^4\mathbf{x}) = \left\{ \underset{\mathbf{x}_1}{\overset{*}{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}}, \underset{\mathbf{x}_2}{\overset{+}{\begin{bmatrix} x_2 \\ x_1 \\ x_0 \\ x_3 \end{bmatrix}}}, \underset{\mathbf{x}_3}{\overset{+}{\begin{bmatrix} x_0 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}}}, \underset{\mathbf{x}_4}{\overset{*}{\begin{bmatrix} x_2 \\ x_3 \\ x_0 \\ x_1 \end{bmatrix}}}, \underset{\mathbf{x}_5}{\overset{+}{\begin{bmatrix} x_1 \\ x_0 \\ x_3 \\ x_2 \end{bmatrix}}}, \underset{\mathbf{x}_6}{\overset{*}{\begin{bmatrix} x_3 \\ x_0 \\ x_1 \\ x_2 \end{bmatrix}}}, \underset{\mathbf{x}_7}{\overset{*}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{bmatrix}}}, \underset{\mathbf{x}_8}{\overset{+}{\begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}}} \right\}$$

with: $$\mathbb{I}_{\mathbb{C}T}({}^4\mathbf{x}) = \overbrace{\mathcal{T}(\mathbf{x})}^{*} \cup \overbrace{\mathcal{T}(\psi(\mathbf{x}))}^{+}$$

Union of all cyclic permutations of x and all cyclic permutations of the reflected pattern $\psi(\mathbf{x})$.