The two-dimensional translation invariant transformation $\mathbb{C}T_{2D}$

The two-dimensional translation can be factorized into two one-dimensional translations with respect to all rows and columns .

$$\tau_{ij}(\mathbf{X}) = \tau_{c_i} \circ \tau_{r_j} \circ \mathbf{X} = \tau_{r_j} \circ \tau_{c_i} \circ \mathbf{X}$$

denoting:

τ_{c_i} a cyclic permutation of all columns by i places

τ_{r_j} a cyclic permutation of all rows by j places

From this follows directly: a two-dimensional transformation of the class $\mathbb{C}T_{2D}$ can be retrieved from first one-dimensionally transforming all columns and then all rows $\mathbb{C}T_{\mathrm{zs}}$, or vice versa $\mathbb{C}T_{\mathrm{sz}}$ (indices are to be read from right to left!):

$$\mathbb{C}T_{rc}(\tau_{ij}(\mathbf{X})) = \mathbb{C}T_{rc}(\mathbf{X})$$

and also:

$$\mathbb{C}T_{cr}(\tau_{ij}(\mathbf{X})) = \mathbb{C}T_{cr}(\mathbf{X})$$

Proof:

$$\mathbb{C}T_{rc}(\tau_{ij}(\mathbf{X})) = \mathbb{C}T_r \circ \underbrace{\mathbb{C}T_c \circ \tau_{c_i}}_{=\mathbb{C}T_c \text{ (1D-Inv.)}} \circ \tau_{r_j} \circ \mathbf{X} = \mathbb{C}T_r \circ \underbrace{\mathbb{C}T_c \circ \tau_{r_j}}_{\text{commutative}} \circ \mathbf{X}$$

$$= \underbrace{\mathbb{C}T_r \circ \tau_{r_j}}_{\mathbb{C}T_r} \circ \mathbb{C}T_c \circ \mathbf{X} = \mathbb{C}T_r \circ \mathbb{C}T_c \circ \mathbf{X} = \mathbb{C}T_{rc}(\mathbf{X})$$

♣

For $\mathbb{C}T_{\text{sz}}$ the conclusion can be established analogously, but keep the following property in mind:

$$\mathbb{C}T_{rc}(\mathbf{X}) \neq \mathbb{C}T_{cr}(\mathbf{X})$$

Two shifted grey scale patterns

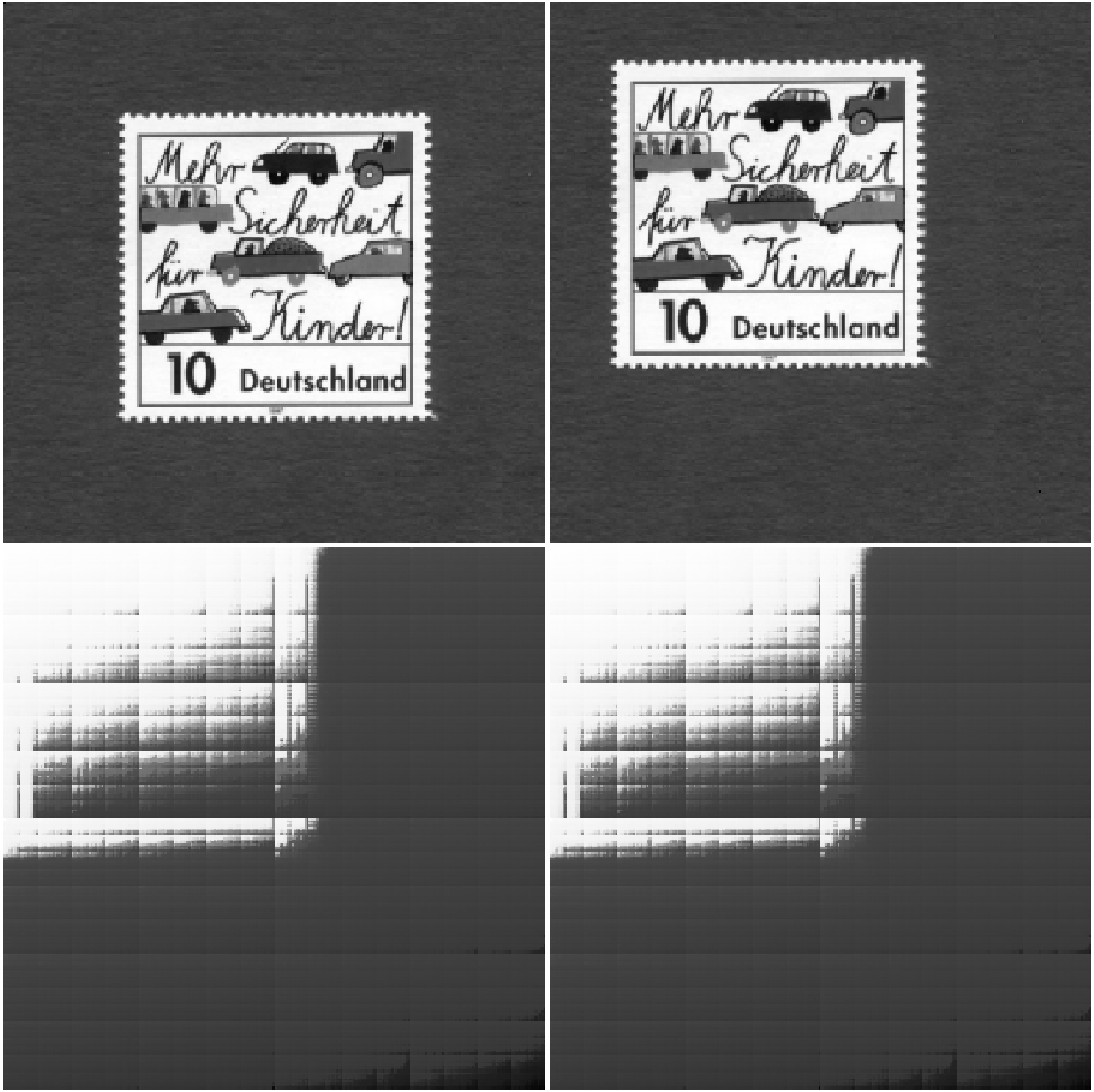
$$\mathbf{X} = \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&2&1&0&0\\ 0&0&1&2&3&2&1&0\\ 0&0&0&0&2&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \qquad \tau_{2,2}(\mathbf{X}) = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&1&2&1&0&0&0&0\\ 1&2&3&2&1&0&0&0\\ 0&0&2&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

Invariant R-transformed

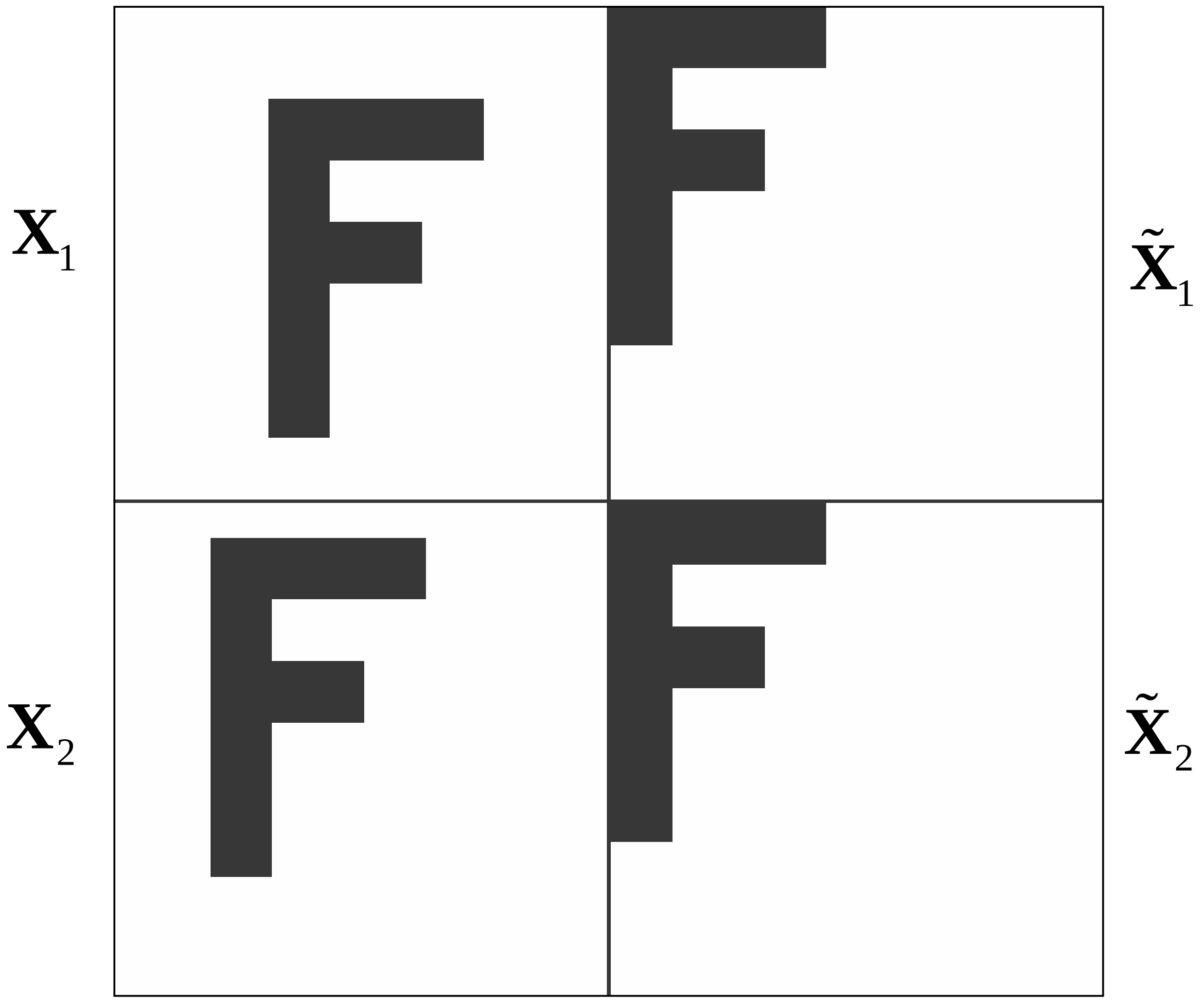
$$\tilde{\mathbf{X}} = \begin{bmatrix} 17 & 5 & 7 & 7 & 15 & 3 & 9 & 9 \\ 5 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 9 & 3 & 1 & 1 & 7 & 5 & 1 & 1 \\ 5 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 15 & 3 & 5 & 5 & 13 & 1 & 7 & 7 \\ 3 & 1 & 3 & 3 & 3 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 9 & 3 & 3 & 3 \\ 7 & 3 & 1 & 1 & 5 & 1 & 3 & 3 \end{bmatrix} \qquad \widetilde{(\tau_{2,2}(\mathbf{X}))} = \begin{bmatrix} 17 & 5 & 7 & 7 & 15 & 3 & 9 & 9 \\ 5 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 9 & 3 & 1 & 1 & 7 & 5 & 1 & 1 \\ 5 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 15 & 3 & 5 & 5 & 13 & 1 & 7 & 7 \\ 3 & 1 & 3 & 3 & 3 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 9 & 3 & 3 & 3 \\ 7 & 3 & 1 & 1 & 5 & 1 & 3 & 3 \end{bmatrix}$$

Invariant B-transformed

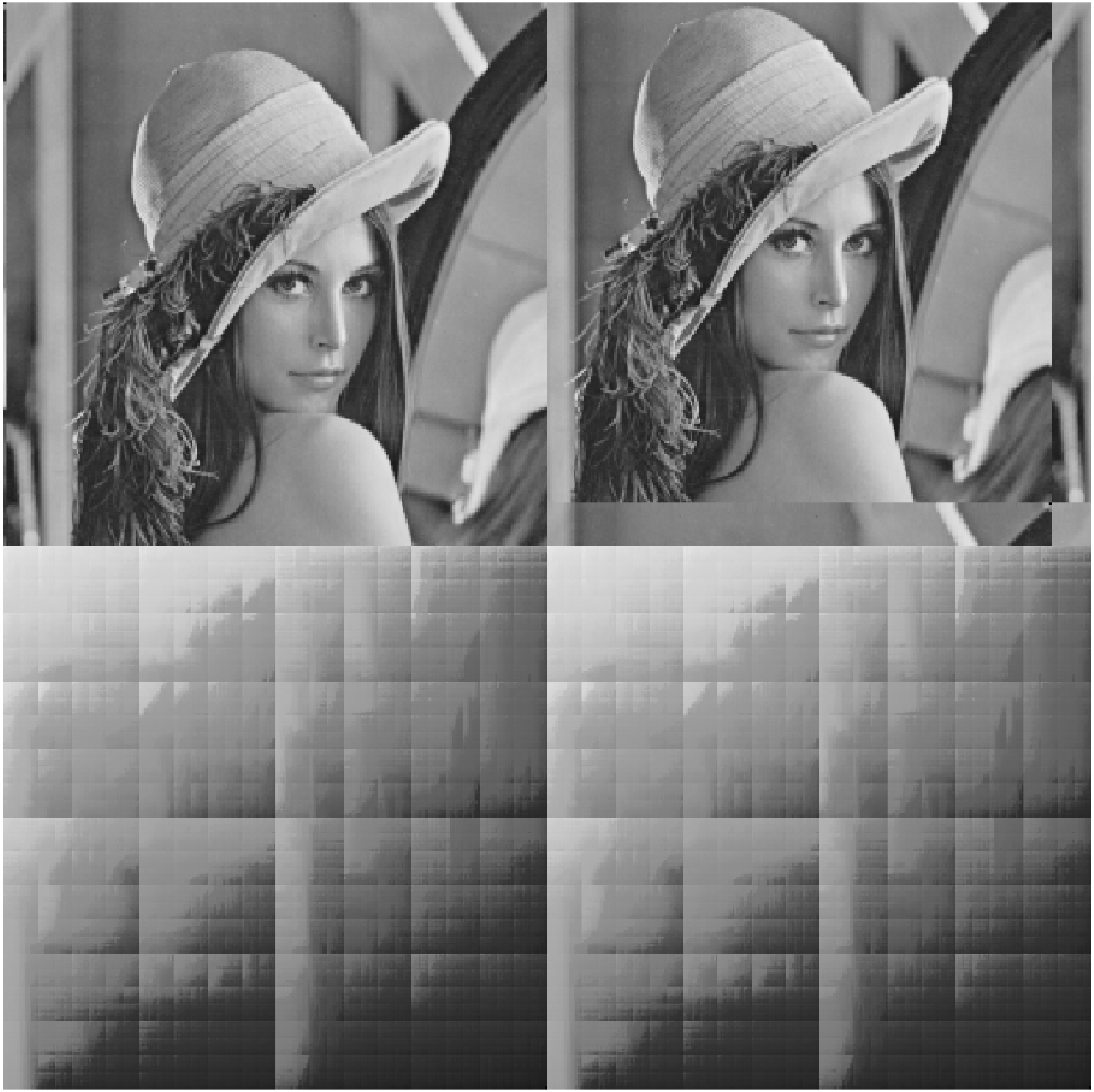
$$\tilde{\mathbf{X}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 & 3 & 3 \end{bmatrix} \qquad \widetilde{(\tau_{2,2}(\mathbf{X}))} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 & 3 & 3 \end{bmatrix}$$



MT_{rc} with homogenous background

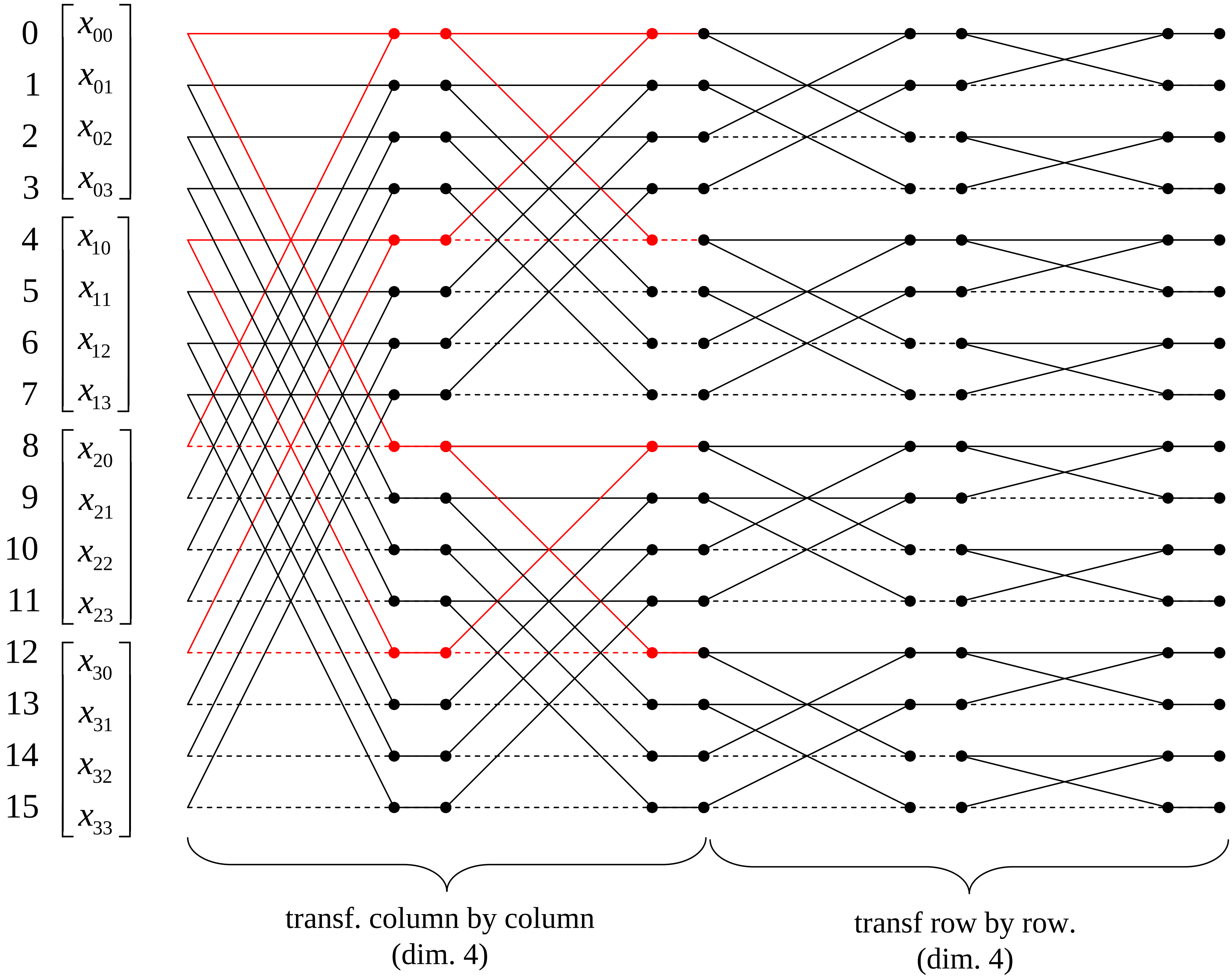


BT_{rc} with homogenous background

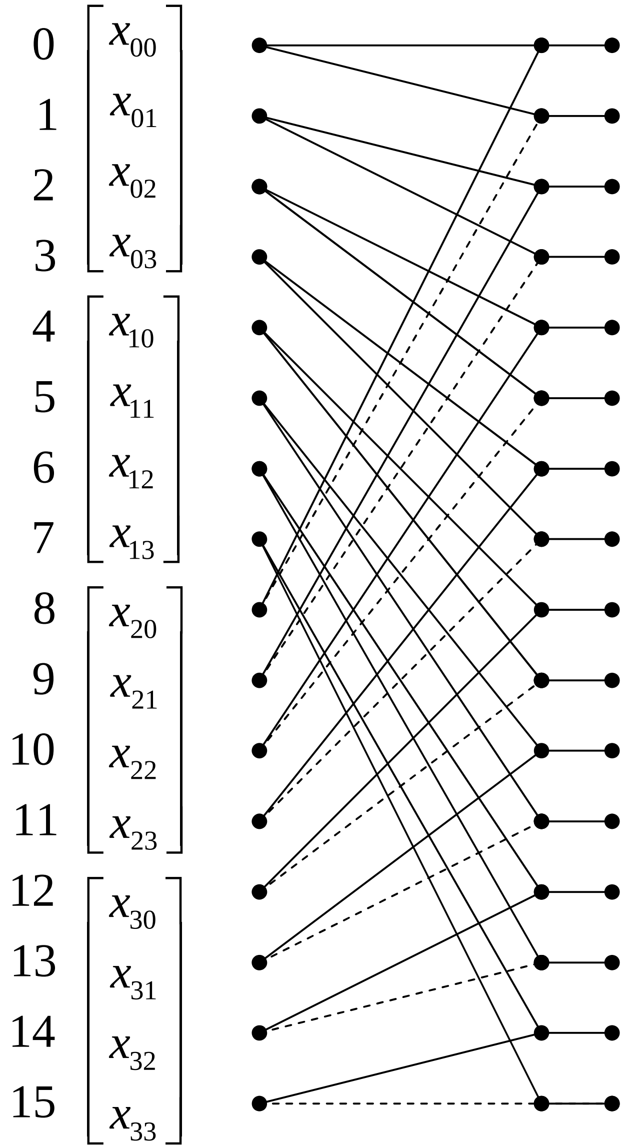


MT_{rc} with inhomogenous background

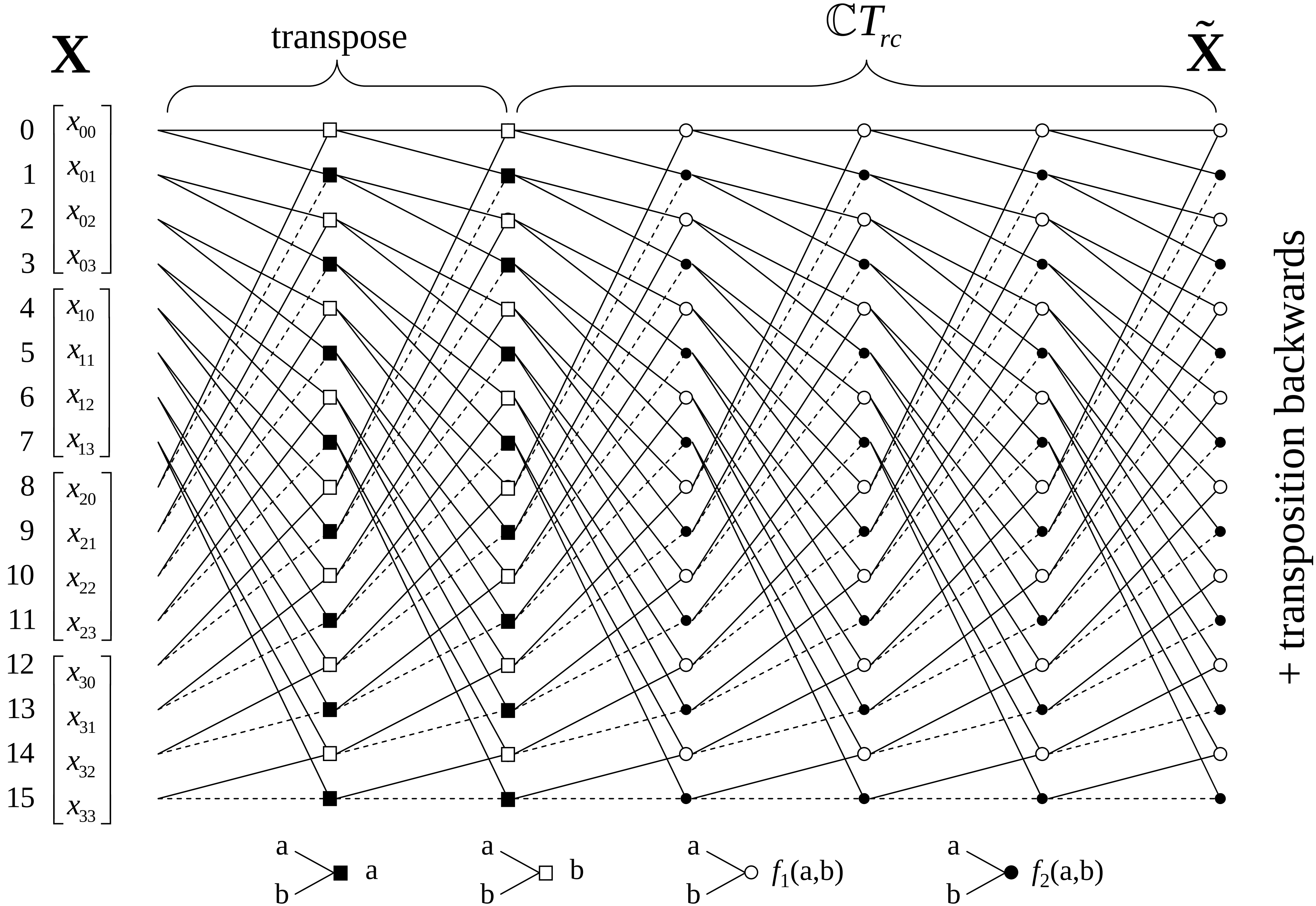
Realizing the two-dimensional $\mathbb{C}T_{rc}$ using one-dimensional transformations



Homogenous structure of the one-dimensional realization of the two-dimensional $\mathbb{C}T_{rc}$



Homogenous structure of the one-dimensional realization of the two-dimensional $\mathbb{C}T_{cr}$



A more general class of the two-dimensional translation invariant transformation $\mathbb{C}T_{2D}$

Obviously applies:

$$\mathbb{C}T_{cr} = \omega_T^{-1} \circ \mathbb{C}T_{rc} \circ \omega_T \circ \mathbf{X}$$

The last transformation can be omitted, because the transposed result is invariant:

$$\mathbb{C}T'_{cr} = \mathbb{C}T_{rc} \circ \omega_T \circ \mathbf{X}$$

The diagonal transform $\mathbb{C}T_{DI}$ belongs to this class

It is defined as:

$$\mathbb{C}T_{DI} = \mathbb{C}T_{rc}(\omega_{DI}(\mathbf{X}))$$

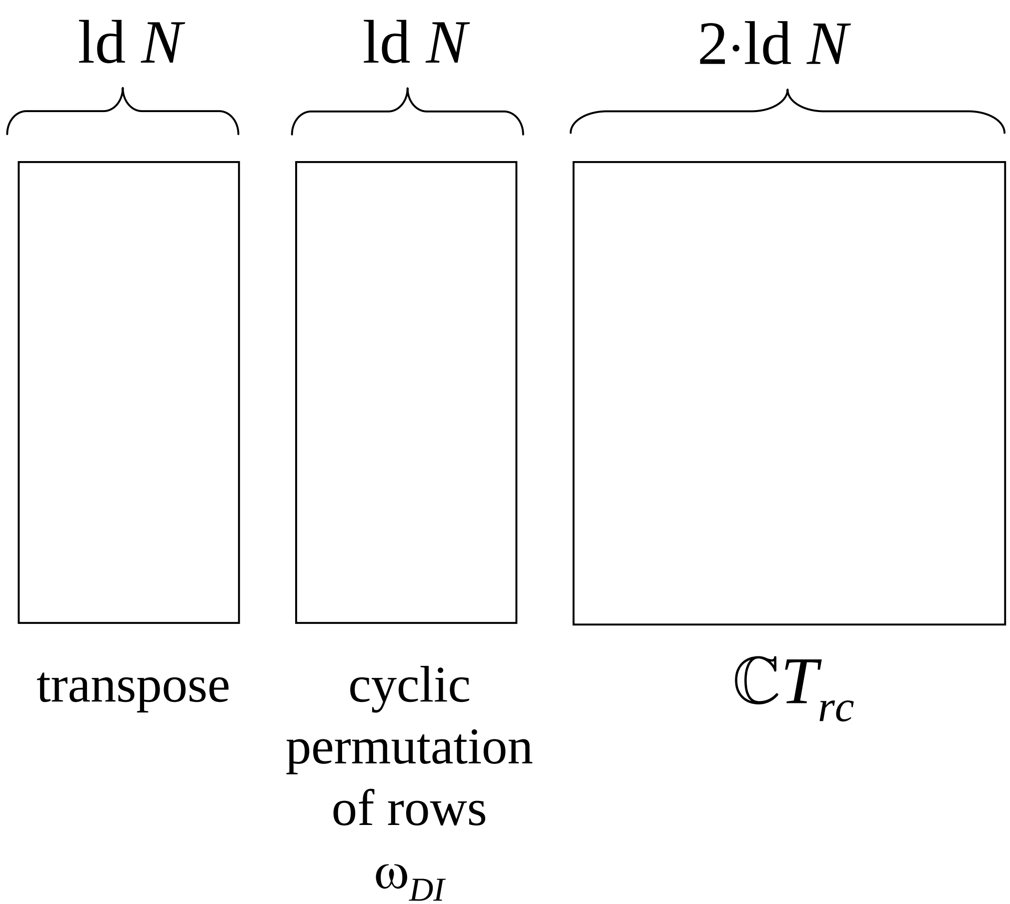
with: $\mathbf{X} = \begin{bmatrix} \mathbf{x}_0^T \\ \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_{N-1}^T \end{bmatrix}$ and $\dim(\mathbf{X}) = N \times N$

and the definition of the diagonal permutation:

$$\omega_{DI}(\mathbf{X}) = \begin{bmatrix} \tau_0(\mathbf{x}_0^T) \\ \tau_1(\mathbf{x}_1^T) \\ \vdots \\ \tau_{N-1}(\mathbf{x}_{N-1}^T) \end{bmatrix} = \begin{bmatrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,N-1} \\ x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{N-1,N-1} & x_{N-1,0} & x_{N-1,1} & \cdots & x_{N-1,N-2} \end{bmatrix}$$

cyclic permutation of k-th row k-times

Implementing the $\mathbb{C}T_{DI}$ on parallel computers with homogenous network



4·ld N passes are needed altogether

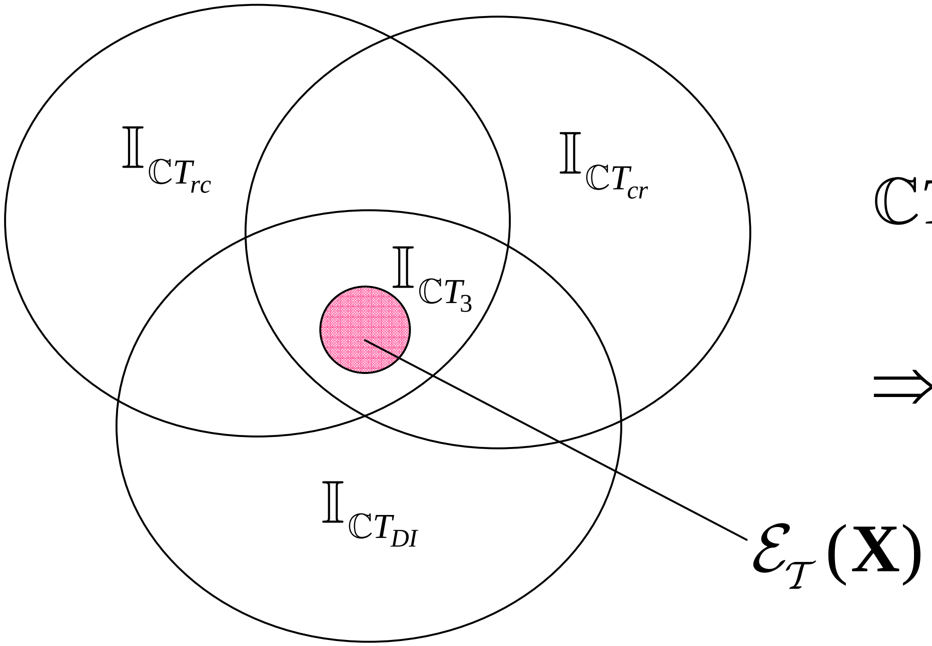
Two-dimensional translation invariant transformation class with a set of *more general compatible permutations* $\Omega := \{\omega_i\}$

$$\mathbb{C}T_{rc}(\omega_k(\tau_{ij}(\mathbf{X})) = \mathbb{C}T_{rc}(\omega_k(\mathbf{X})) \quad \text{for} \quad \forall \omega_k \in \Omega$$

(without proof!)

Increasing the degree of completeness by combining various invariant transforms

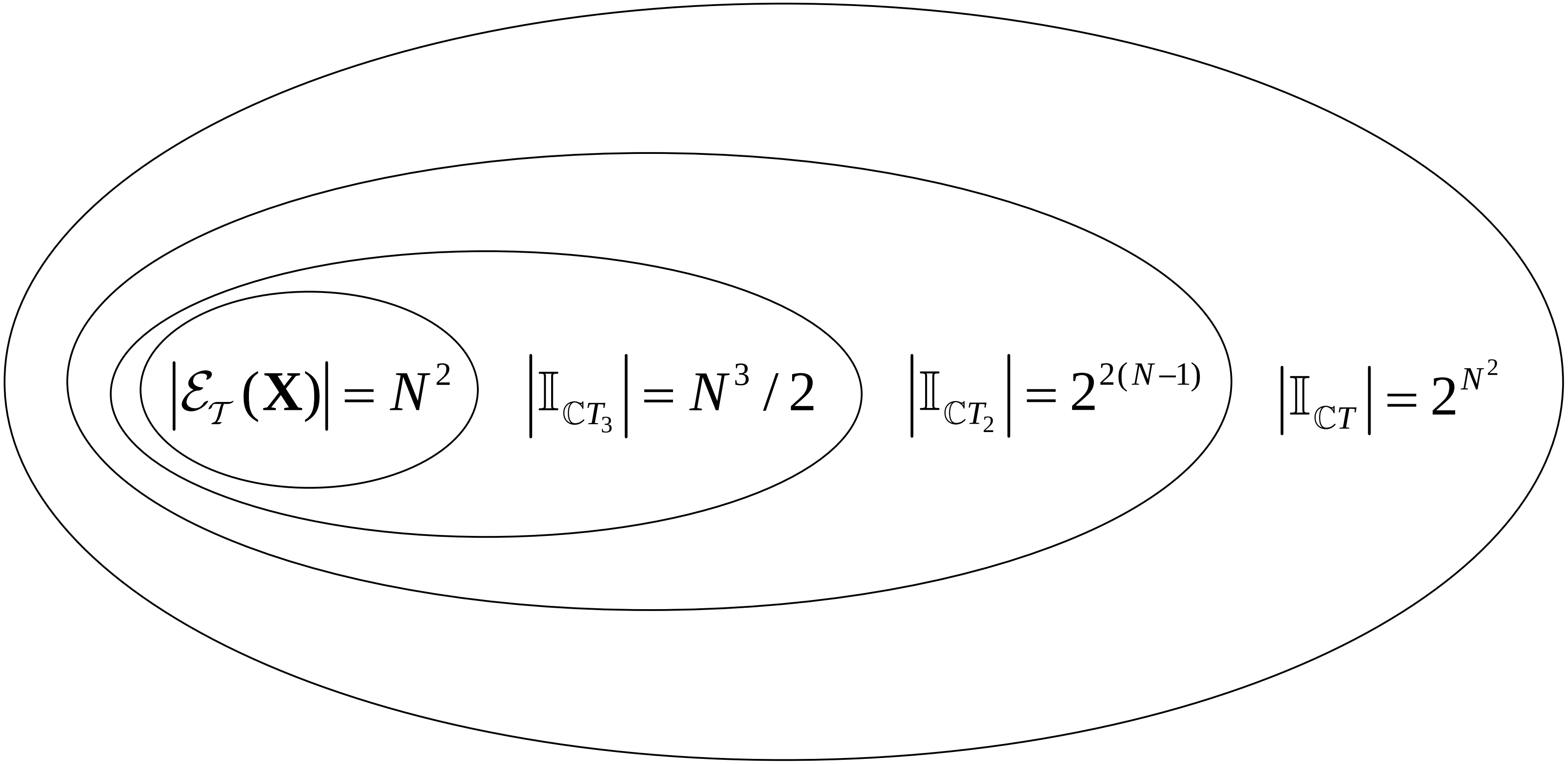
Combining two or more transformations results in an intersection of the original transforms, that consists of the invariants of the new transform, because all sets of invariants are in general distinct from each other:



$$\mathbb{C}T_3 := \underbrace{\mathbb{C}T_{rc} \cup \mathbb{C}T_{cr}}_{\mathbb{C}T_2} \cup \mathbb{C}T_{DI}$$

$$\Rightarrow \mathbb{I}_{\mathbb{C}T_3} := \mathbb{I}_{\mathbb{C}T_{rc}} \cap \mathbb{I}_{\mathbb{C}T_{cr}} \cap \mathbb{I}_{\mathbb{C}T_{DI}}$$

Degree of completeness for the different classes of transformations



Susceptibility of interference of RT

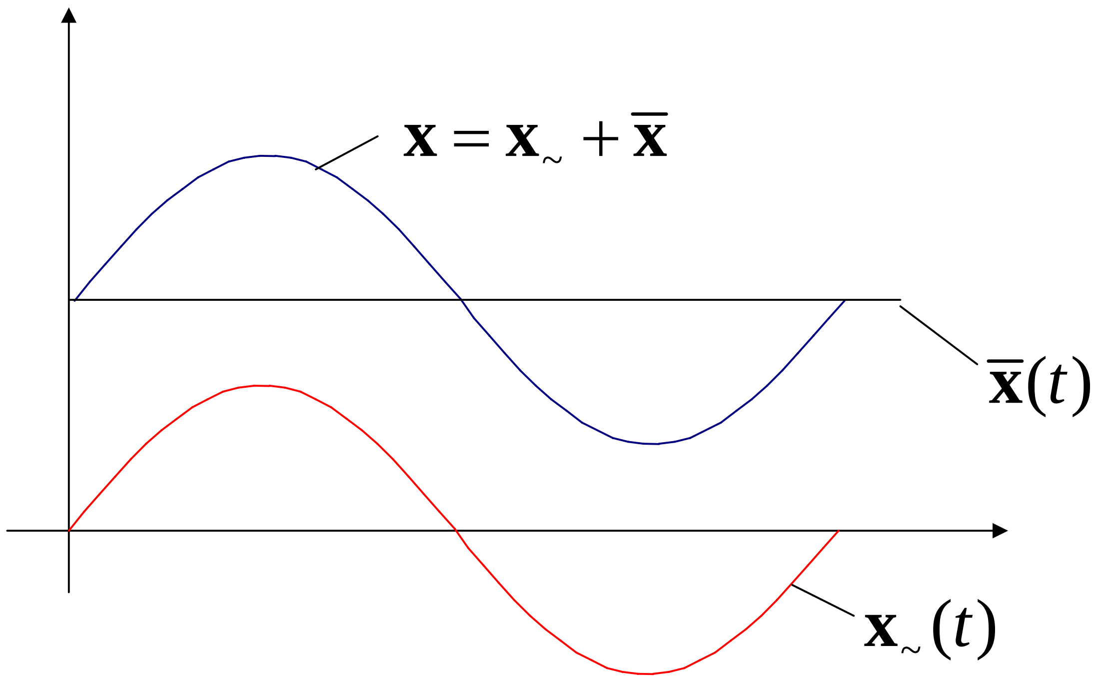
Reaction to systematic noise: changes in brightness and contrast:

$$\mathbf{x}' = k\mathbf{x} + a\mathbf{u} \quad \text{with: } \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Additionally stochastic noise:

$$\mathbf{x}' = \mathbf{x} + \mathbf{n}$$

Dissecting a signal into an alternating and an constant portion



$$\mathbf{x} = \mathbf{x}_{\sim} + \overline{\mathbf{x}} = \underbrace{\mathbf{x}_{\sim}}_{\text{median-free portion}} + \underbrace{\overline{x}\mathbf{u}}_{\text{identical portion}} \quad \text{with: } \overline{x} = \frac{1}{T}\int_{t=0}^{T} x(t)dt \text{ and: } \frac{1}{N}\sum_{i=1}^{N} x_i$$

Sentence: An additional constant portion has only influence on the 0-th coefficient of the RT; ignoring the 0-th coefficient results in invariance

$$\widetilde{(\mathbf{x}+a\mathbf{u})} = \tilde{\mathbf{x}} + a \cdot N \cdot \mathbf{e}_1 = \tilde{\mathbf{x}} + a \cdot N \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad a \in \mathbb{R}$$

$\mathbf{u} \mathrel{\hat{=}}$ constant component, $\mathbf{e}_1 \mathrel{\hat{=}}$ 1. unit vector

Sentence: RT is strictly homogenous wrt. *mean-free* pattern $\mathbf{x}_{\sim}$:

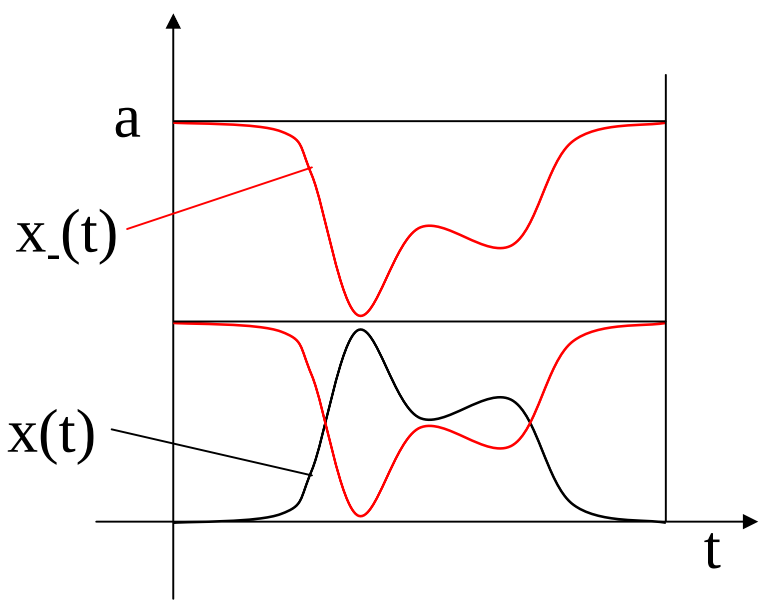
$$\widetilde{k\mathbf{x}_{\sim}} = |k| \widetilde{\mathbf{x}_{\sim}}$$

Proof: complete induction

Inverting the contrast with RT („negative“)

RT is invariant wrt. inverting the contrast, if the 0-th coefficient is suppressed/omitted (which has influence simply on identical portion) :

$$\mathbf{x}_- = a\mathbf{u} - \mathbf{x} = a\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} - \mathbf{x}$$



$$\mathbf{x}_- = a\mathbf{u} - \mathbf{x} = a\mathbf{u} - (\mathbf{x}_\sim + \bar{x}\mathbf{u})$$
$$= (-\mathbf{x}_\sim) + (a - \bar{x})\mathbf{u}$$
$$\Rightarrow \widetilde{\mathbf{x}_-} = \widetilde{\mathbf{x}_\sim} + N(a - \bar{x})\mathbf{e}_1$$

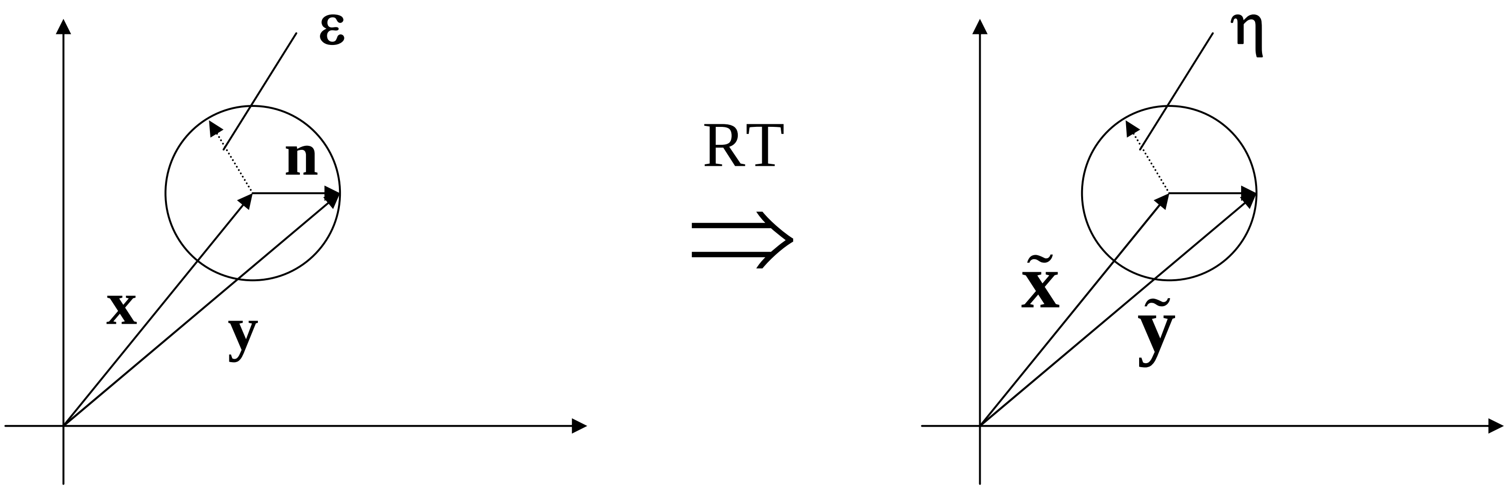
because:

$$\widetilde{(-\mathbf{x}_\sim)} = \widetilde{\mathbf{x}_\sim} \quad \text{and} \quad \widetilde{c\mathbf{u}} = Nc\mathbf{e}_1$$

Assuming that only positive intesities are allowed, the following applies: $a \geq \max(x_i)$

Reaction of RT to stochastic noise

$$\mathbf{x}' = \mathbf{x} + \mathbf{n}$$



The map is continous wrt. to Euclidian metrics!

$$\|\mathbf{y} - \mathbf{x}\| = \|\mathbf{n}\| < \varepsilon \quad \Rightarrow \quad \|\tilde{\mathbf{y}} - \tilde{\mathbf{x}}\| < \eta$$

more precise: $\|\tilde{\mathbf{y}} - \tilde{\mathbf{x}}\| \leq \sqrt{N}\,\|\mathbf{y} - \mathbf{x}\|$ with: $\dim(\mathbf{x}) = \dim(\mathbf{y}) = N$

i.e. $\boxed{\eta = \sqrt{N}\varepsilon}$

Proof:
compl. ind.

Two examples for RT

a) result is on the edge of estimation:

$$\mathbf{x}^T = \begin{bmatrix}1 & 3 & 5 & 6 & 8 & 7 & 4 & 2\end{bmatrix} \quad \mathbf{n}^T = \begin{bmatrix}0,385 & 0,196 & 0,22 & -0,595 & -0,307 & -0,296 & 0,462 & 0,121\end{bmatrix}$$

$$\tilde{\mathbf{x}}^T = \begin{bmatrix}36 & 0 & 2 & 2 & 16 & 0 & 6 & 6\end{bmatrix} \quad \tilde{\mathbf{y}}^T = \widetilde{(\mathbf{x}+\mathbf{n})}^T = \begin{bmatrix}36,2 & 1,33 & 2,98 & 1,77 & 13,9 & 0,274 & 5,77 & 5,33\end{bmatrix}$$

$$\left\|\tilde{\mathbf{y}} - \tilde{\mathbf{x}}\right\| = 2,83 \underbrace{\left\|\mathbf{n}\right\|}_{=1} = \sqrt{N} = \sqrt{8}$$

b) result is inside the estimation:

$$\mathbf{x}^T = \begin{bmatrix}8 & 3 & 5 & 1\end{bmatrix} \quad \mathbf{n}^T = \begin{bmatrix}-4,5 & 4,5 & -4,5 & 4,5\end{bmatrix}$$

$$\tilde{\mathbf{x}}^T = \tilde{\mathbf{y}}^T = \begin{bmatrix}17 & 9 & 5 & 1\end{bmatrix}$$

$$\left\|\tilde{\mathbf{y}} - \tilde{\mathbf{x}}\right\| = 0 \le 2 \underbrace{\left\|\mathbf{n}\right\|}_{=9} = 18$$

Dynamic perimeter of the class $\mathbb{C}T$

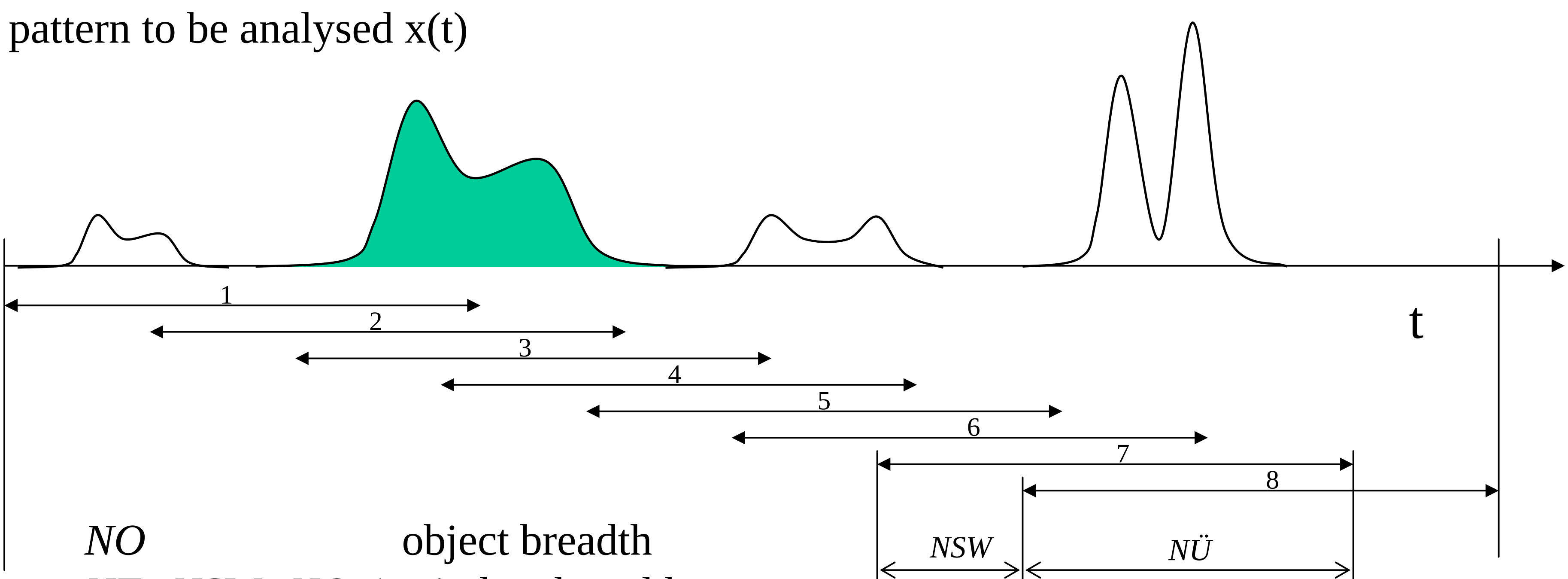
- The B- and the M-transforms are closed, which means that all intermediate and the result are within the range of the A/D converter

- With RT this dymanic perimeter (number of binary posititions) of the transfomed results:

$$\underbrace{m}_{A/D-converter} + n = m + \log_2 N$$

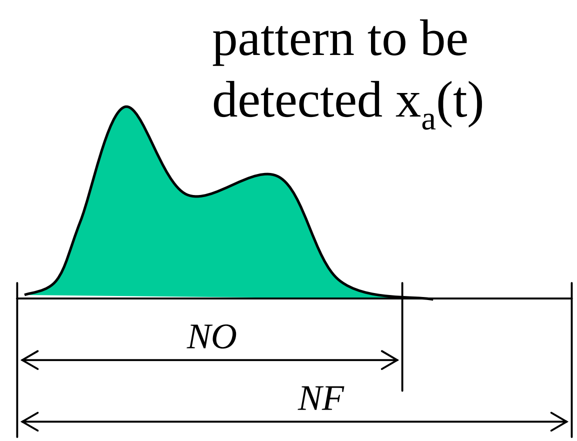
which means that in every layer of the RT maximal one digit is added (doubling the value)

Search (field) strategy with overlapping windows processing one-dimensional signals as an alternative to complete correlation



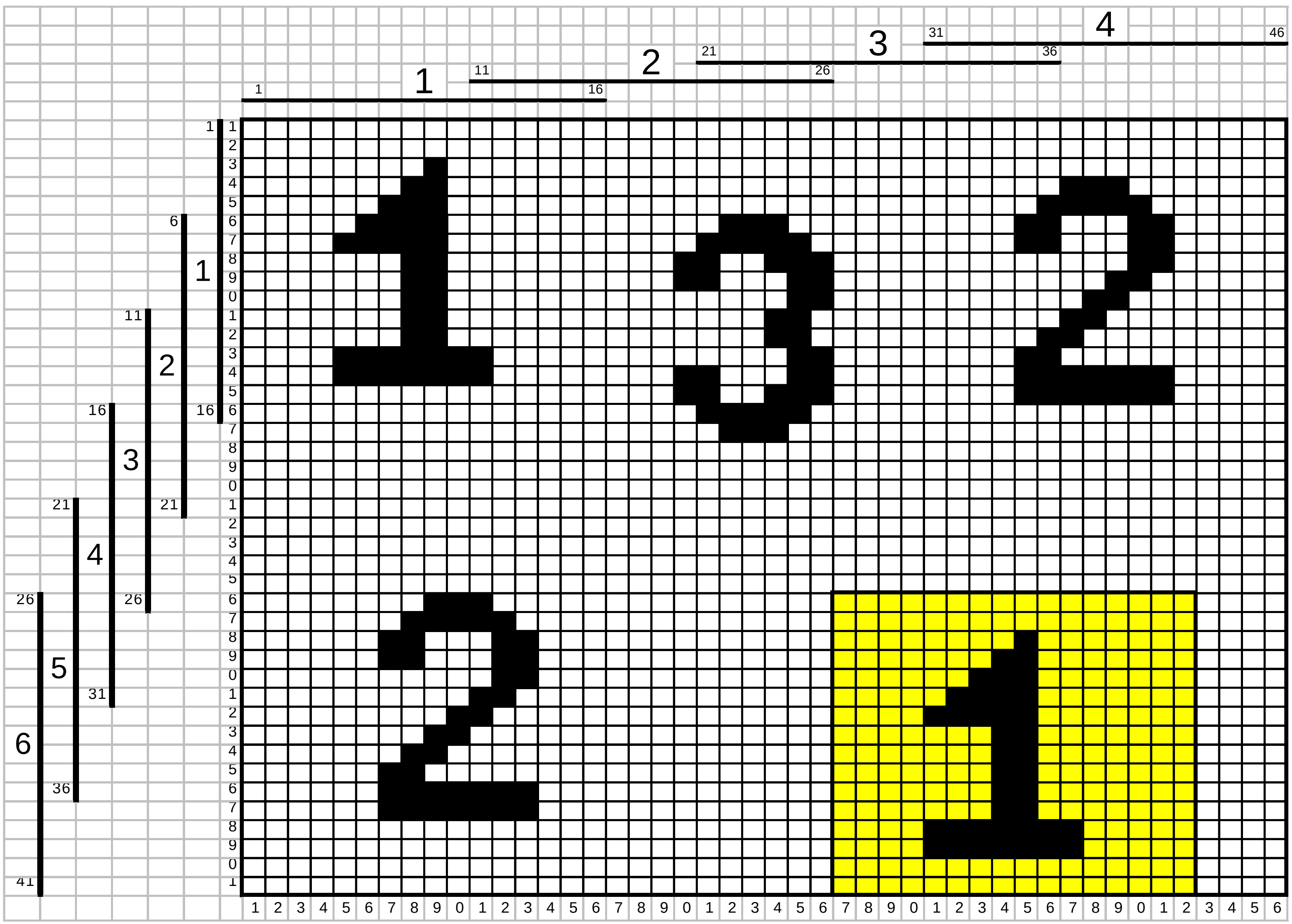
NO object breadth
NF=NSW+NO-1 window breadth
NSW=NF-NÜ step breadth
NÜ=NO-1 overlap

it must be assured, that the object is contained completely in one of the windows!!

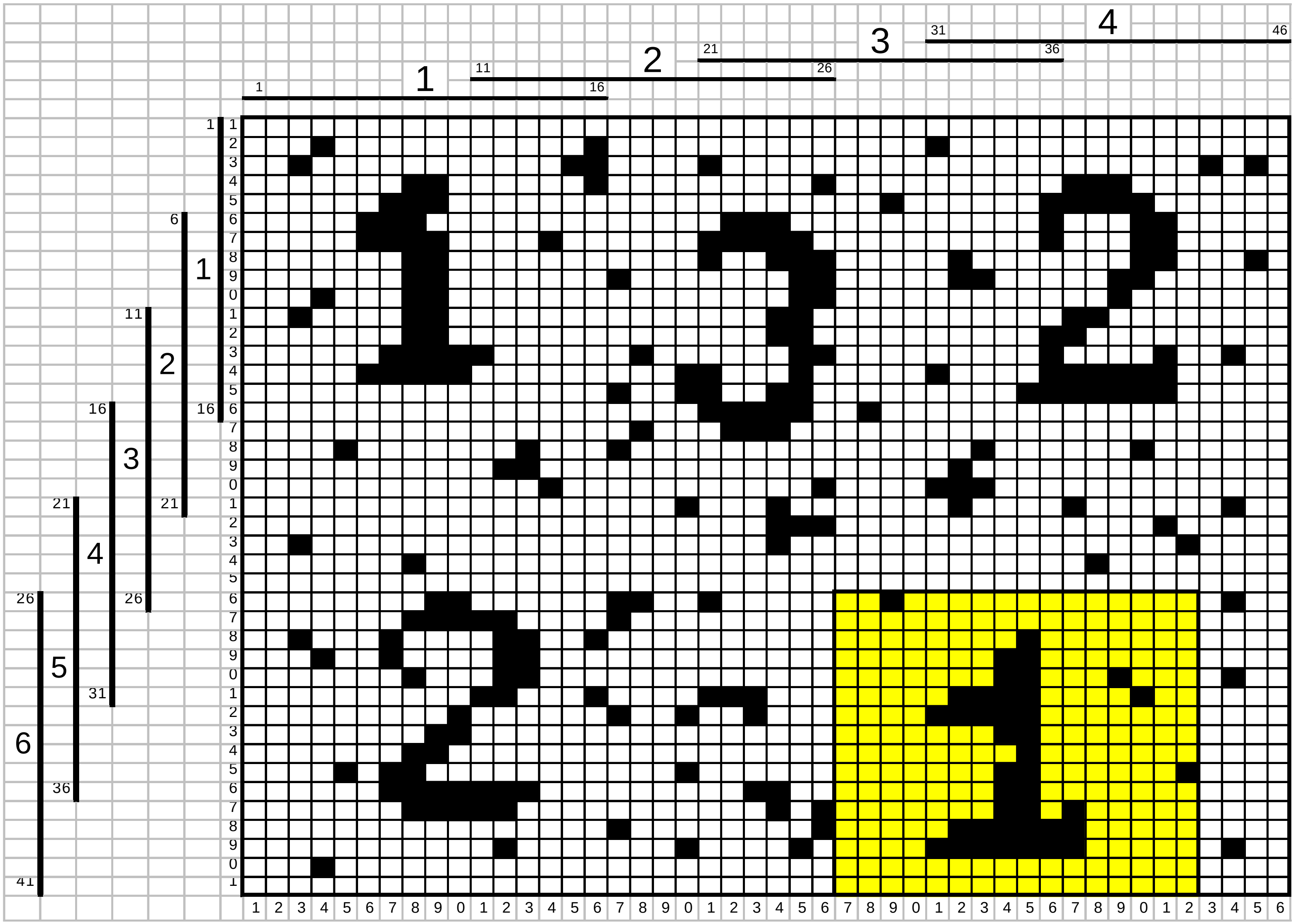


compromise between step breadth and signal/niose correlation (Vereinzelungsbedingung)

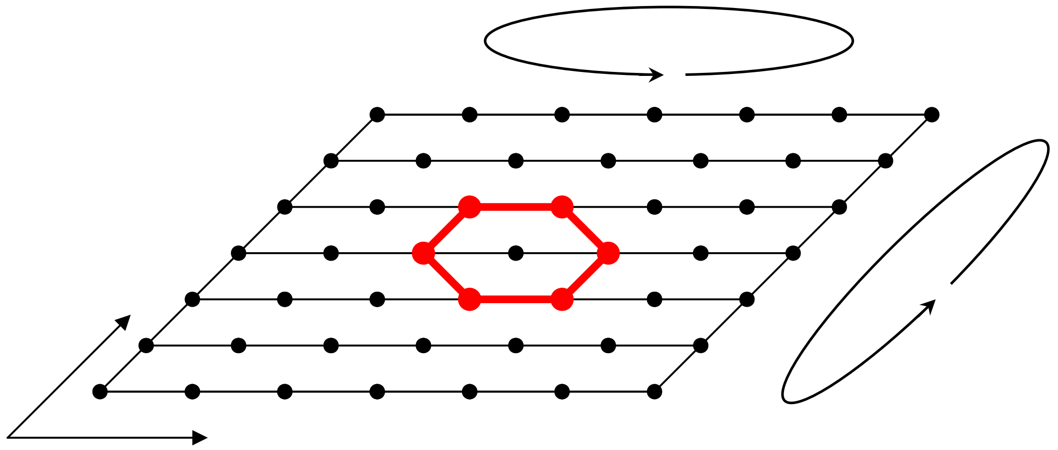
Search (field) strategy with overlapping windows - undisturbed patterns



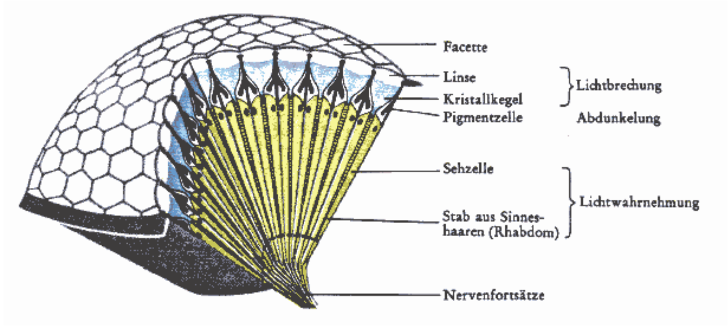
Search (field) strategy with overlapping windows - disturbed patterns



Translation invariance of a hexagonal scan



Insects’ eye



Every translation within a hexagonal pattern can be represented as overlap of two inclined (non-rectangular) base vectors. The base-field is a rhombus.

By reorganizing the data a rectangular pattern can be approximated.