Kapitel 5c

Berechnung von Invarianten für diskrete Objekte

Invariants for Discrete Structures – An Extension of Haar Integrals over Transformation Groups to Dirac Delta Functions

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In C. E. Rasmussen and H. H. Bülthoff and M. A. Giese and B. Schölkopf, editors *Proceedings of the 26th DAGM Symposium,* Tübingen, Germany, Aug./Sep. 2004.

Summary

1. Introduction
2. Invariants for continuous objects
3. Invariants for discrete objects
    - Invariants for polygons
    - 3D-meshes
    - Discrimination performance and completeness
4. Experiments: Object classification in a Tangram database
5. Conclusions

Introduction

- Increased interest in 3D models and 3D sensors induce a growing need to support e.g. the automatic search in such databases
- As the description of 3D objects is not canonical ➔ use invariants for their description

Invariant integration over Euclidean group

For (cyclic) image translation and rotation:

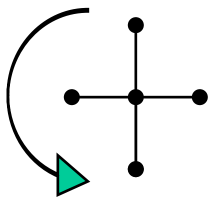
$$(g\mathbf{X})[i, j] = \mathbf{X}[k, l]$$

$$\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} - \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}$$

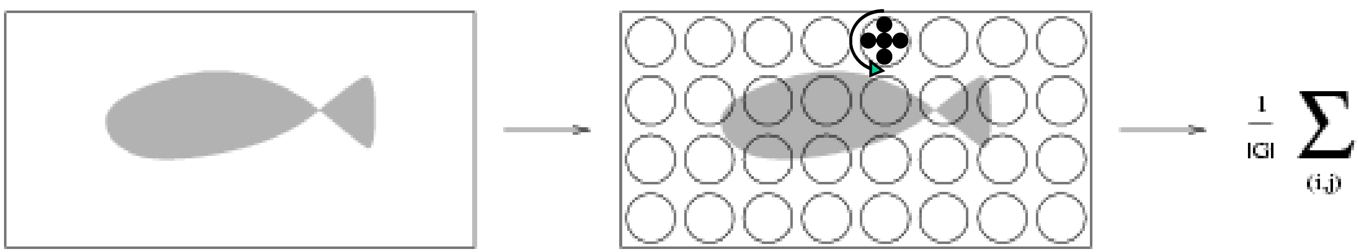
all indices to be understood modulo the image dimensions.

$$A[f](\mathbf{X}) = \frac{1}{2\pi NM} \int_{t_0=0}^{N} \int_{t_1=0}^{M} \int_{\varphi=0}^{2\pi} f(g\mathbf{X})\, d\varphi\, dt_1\, dt_0$$

Use as kernel functions monomials of pixels of local support and integrate over the Euclidean motion:



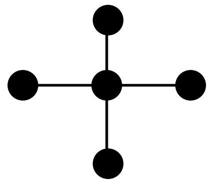
$$f(\mathbf{X}) = m_{00}^{3} m_{01}^{1} m_{0-1}^{5} m_{10}^{2} m_{-10}^{3}$$



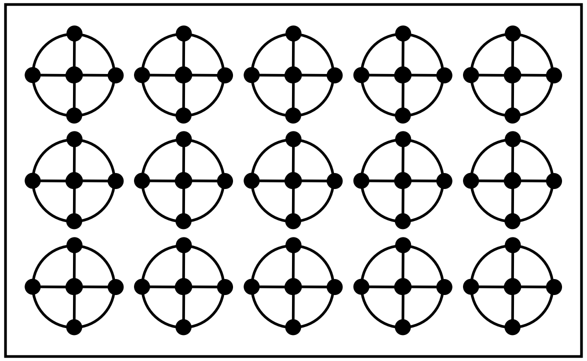
Image

Evaluation of a local function for each pixel of the image

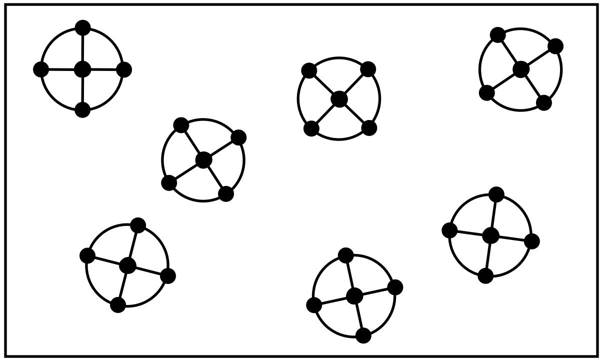
Sum over all these local results



Monomial

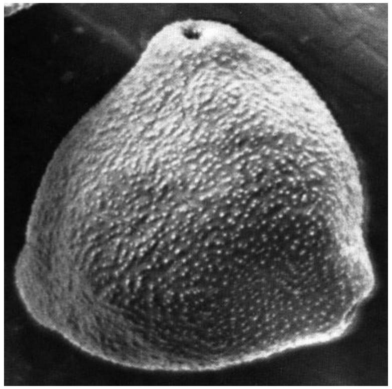


Deterministic integral
over the planar
Euclidean motion

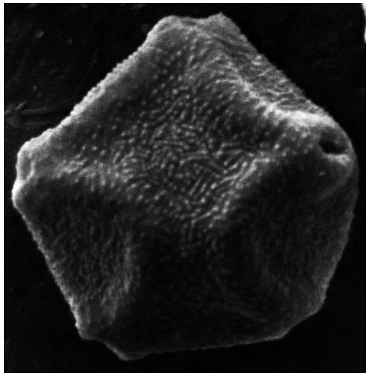


Monte-Carlo-Integration

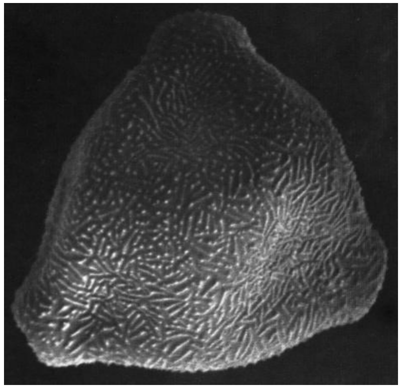
Pollen examples



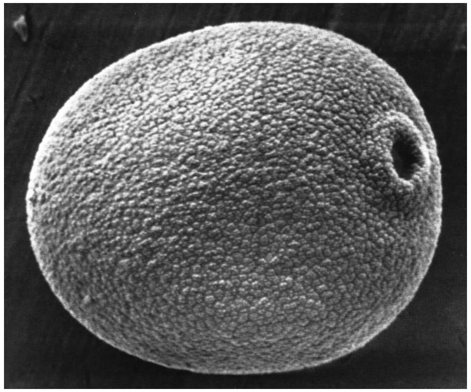
Hasel



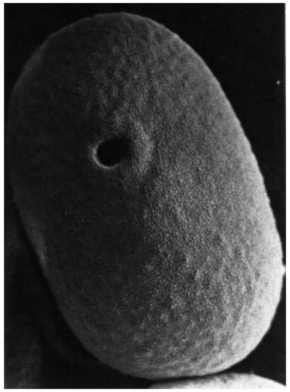
Erle



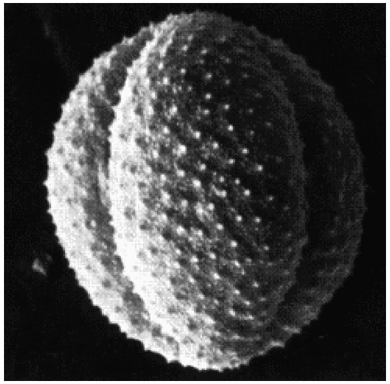
Birke



Gräser



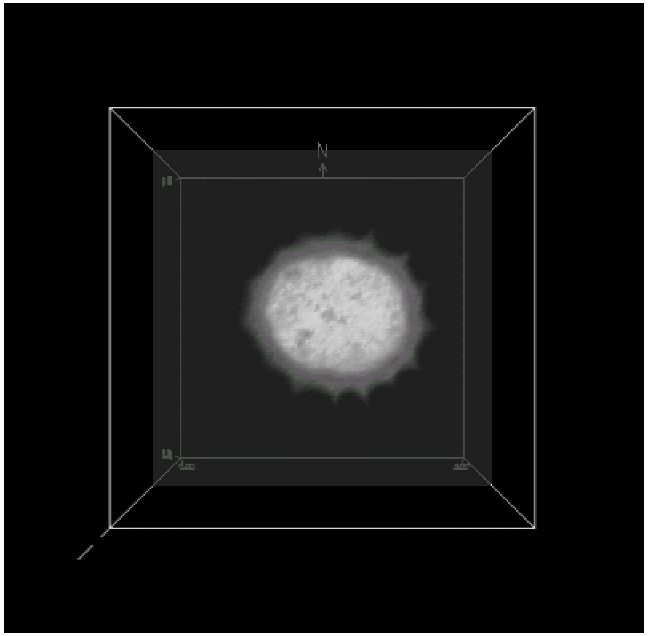
Roggen



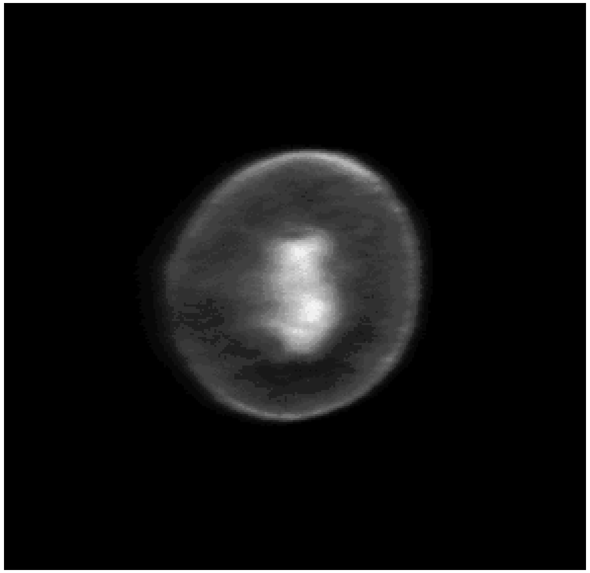
Beifuß

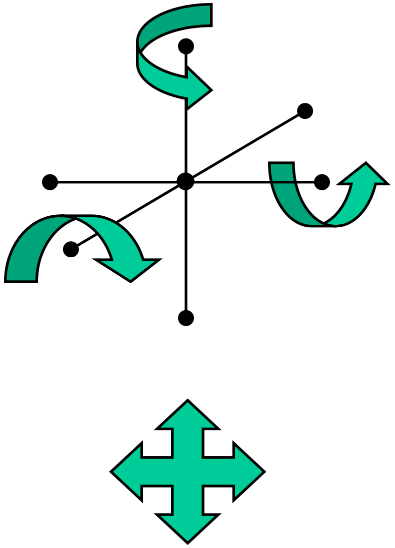
+ 33 further species (not relevant for allergies)

Gänseblümchen/daisy pollen grain



Eibe/Taxus





Integrate over Euclidean Motion

Extension of Haar-Integrals to Discrete Structures

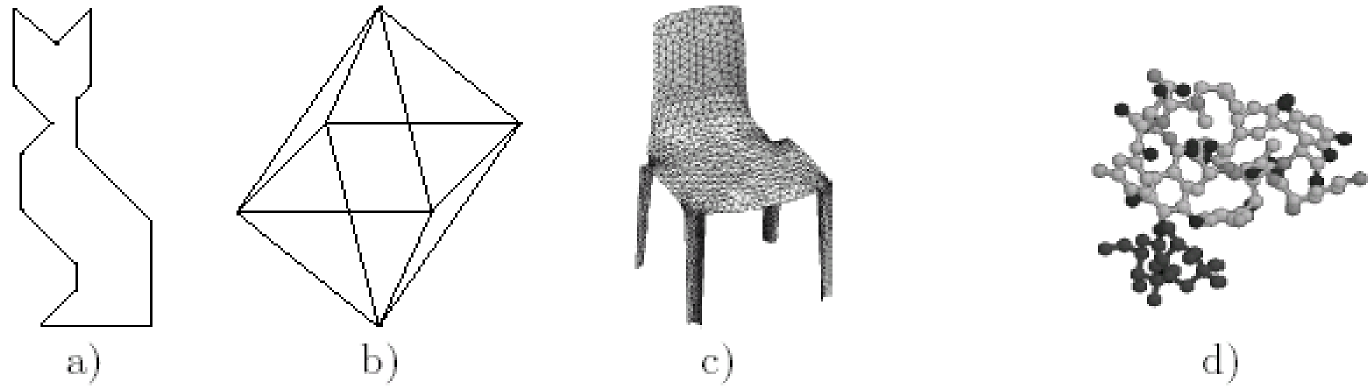
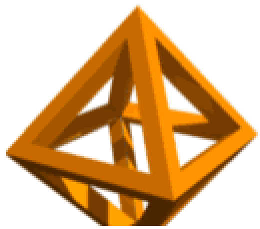


Fig. 1. Discrete structures in 2D and 3D: (a) closed contour described by a polygon (b) wireframe object (c) 3D triangulated surface mesh (d) molecule.

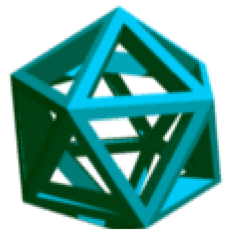
Describe discrete structures with Dirac delta functions!

The Five Platonic Solids

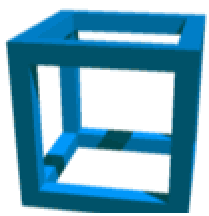
Octahedron



Icosahedron.gif



Dodecahedron



Hexahedron

Tetrahedron

A platonic solid is a polyhedron (Polyeder) all of whose faces are congruent (they differ only in a Euclidean motion) regular polygons, and where the same number of faces meet at every vertex. The best know example is a *cube* (or *hexahedron*) whose faces are six congruent squares.

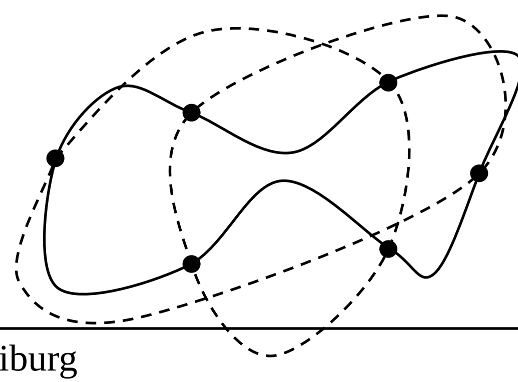
Invariants on discrete Structures

(topologically equivalent structures)

Chose proper kernel functions on distributions

$$T \quad f \quad \mathbf{X} = \int_G f(g\mathbf{X})\,dg$$

1. Chose kernel functions which are different from zero only at the vertices and which act only on neighborhoods of degree *m*.
2. As each vertex can be visited in an arbitrary permutation of all points by a continuous Euclidean motion the integral is changed into a invariant sum over all vertices with Euclidean-invariant local discrete features !!
3. Use principle of rigidity to reach completeness: use a basis of features which are locally rigid and which can be pieced together in a unique way to the global object (see invariants for triangle!).



Invariants for discrete objects

1. For a discrete object **Δ** and a kernel function $f(\boldsymbol{\Delta})$ it is possible to construct an invariant feature $T[f](\boldsymbol{\Delta})$ by integrating $f(g\boldsymbol{\Delta})$ over the Euclidean transformation group $g \in G$.
2. The kernel function is properly designed, such that it delivers a value dependent on the discrete features of a local neighborhood, when a vertex of the object moved by the continuous Euclidean motion g hits the origin and has one specific orientation.
3. Let us assume that our discrete object is different from zero only at its vertices. A rotation and translation invariant local discrete kernel function h takes care for the algebraic relations to the neighboring vertices and we can write:

$$f \quad \Delta \quad = \sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i)\,\delta(\mathbf{x} - \mathbf{x}_i)$$

where $\mathbb{V}$ is the set of vertices and $\mathbf{x}_i$ the vector representing vertex i.

4. In order to get finite values from the distributions it is necessary to introduce under the Haar integral another integration over the spatial domain **X**.
5. By choosing an arbitrary integration path in the continuous group G we can visit each vertex in an arbitrary order the integral is transformed into a sum over all local discrete functions allowing all possible permutations of the contributions of the vertices.

Extension of Haar-Integrals to Discrete Structures

$$T \quad f \quad \Delta := \int_G \int_{\mathbf{X}} f(g\Delta) d\mathbf{x} dg = \int_G \left[\int_{\mathbf{X}} \sum_{i \in \mathbb{V}} h(g\Delta, g\mathbf{x}_i) \delta(g\mathbf{x} - g\mathbf{x}_i) d\mathbf{x} \right] dg$$

$$= \int_G \left[\sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i) \right] dg = \sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i)$$

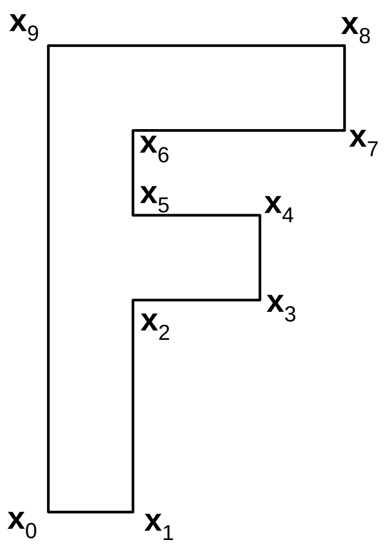
Intuitive result: get *global* Euclidean invariants by summation over discrete *local* Euclidean invariants $h(\boldsymbol{\Delta}, \mathbf{x}_\mathrm{i})$!

Remember: The delta function has the following selection property:

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$$

Euclidean Invariants for Polygons

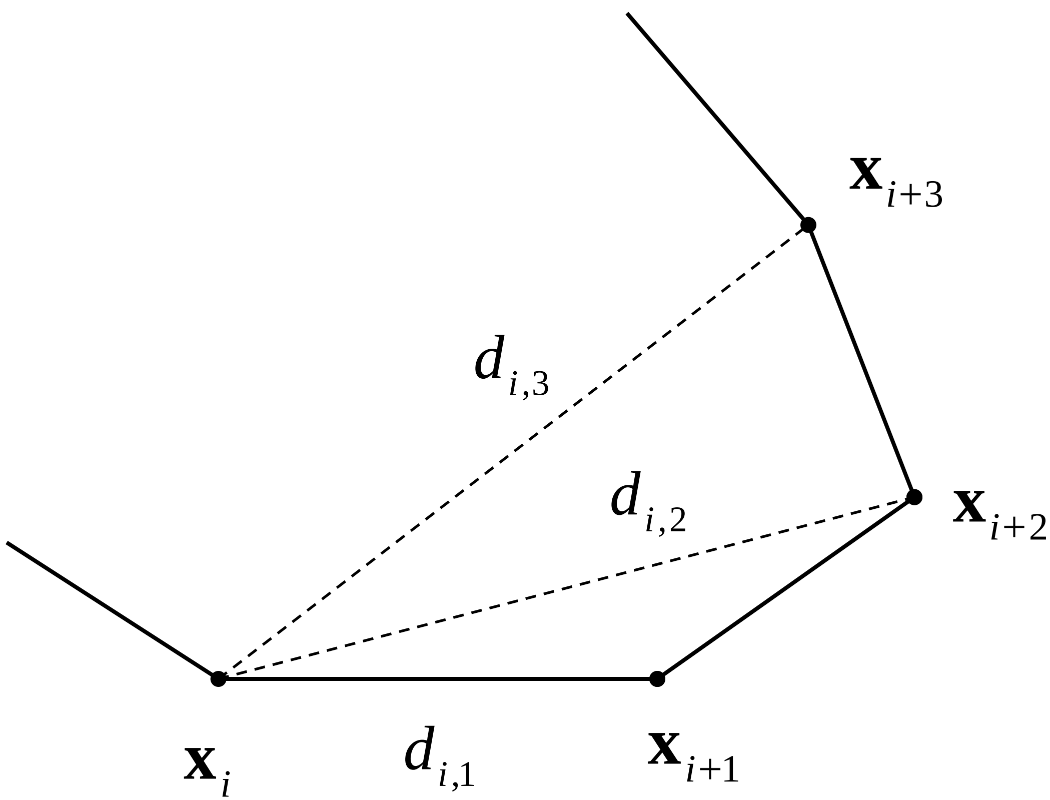
We assume e.g. to have given a polygon with 10 vertices, e.g.



$$\mathbf{x}_i = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 1 & 1 & 2.5 & 2.5 & 1 & 1 & 3.5 & 3.5 & 0 \\ 0 & 0 & 2.5 & 2.5 & 3.5 & 3.5 & 4.5 & 4.5 & 5.5 & 5.5 \end{array}\right]$$

Choose as local Euclidean invariants distances of vertex i and its k-th righthand neighbours:

$$d_{i,k} = \left\| \mathbf{x}_i - \mathbf{x}_{<i+k>} \right\|$$

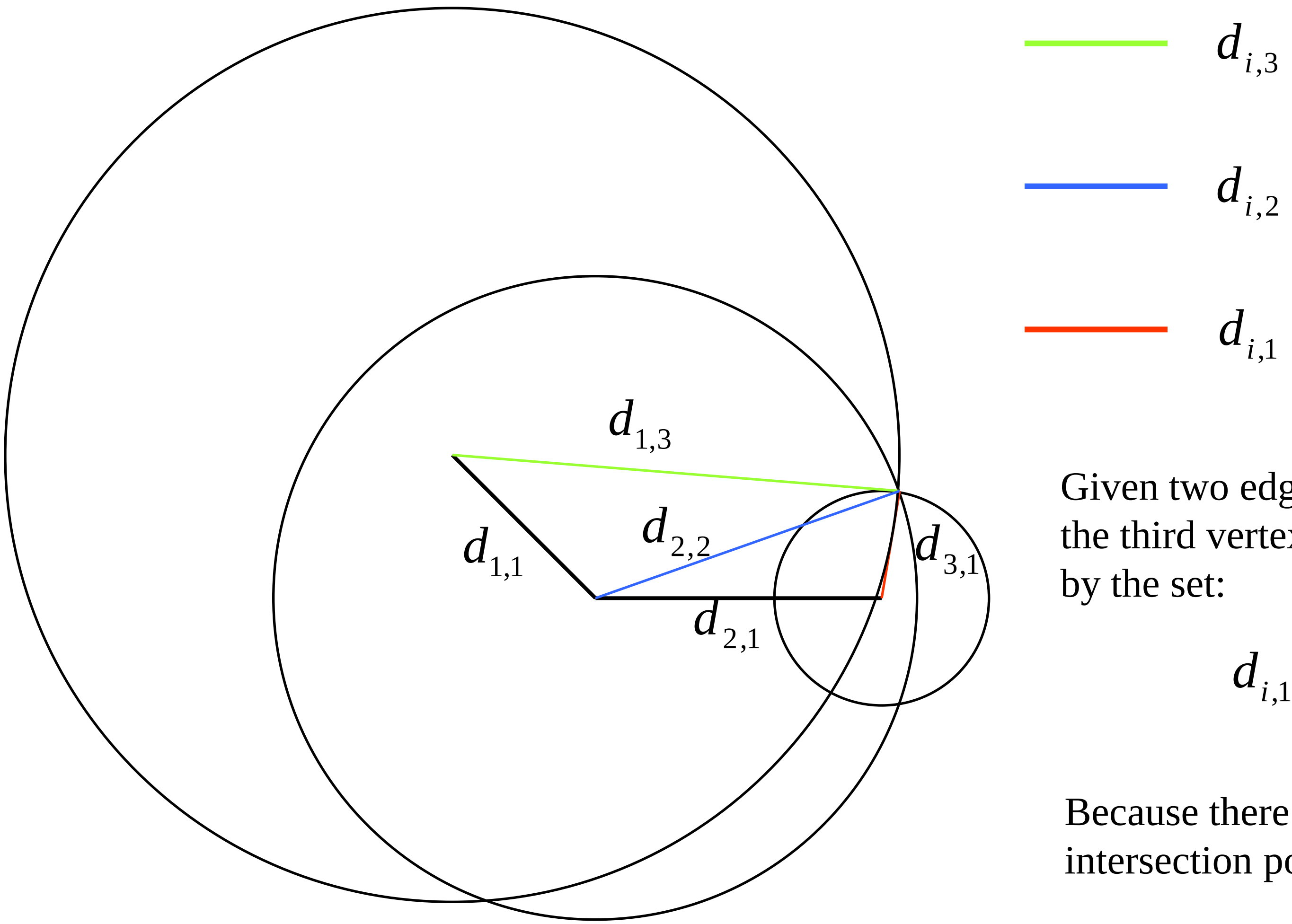


The elements:

$$d_{i,1}, d_{i,2}, d_{i,3}$$

form a basis for a polygon, because they uniquely define a polygon (up to a mirror-polygon) !

Principle of rigidity



Given two edges $d_{1,1}$ and $d_{2,1}$. Then the third vertex is uniquely defined by the set:

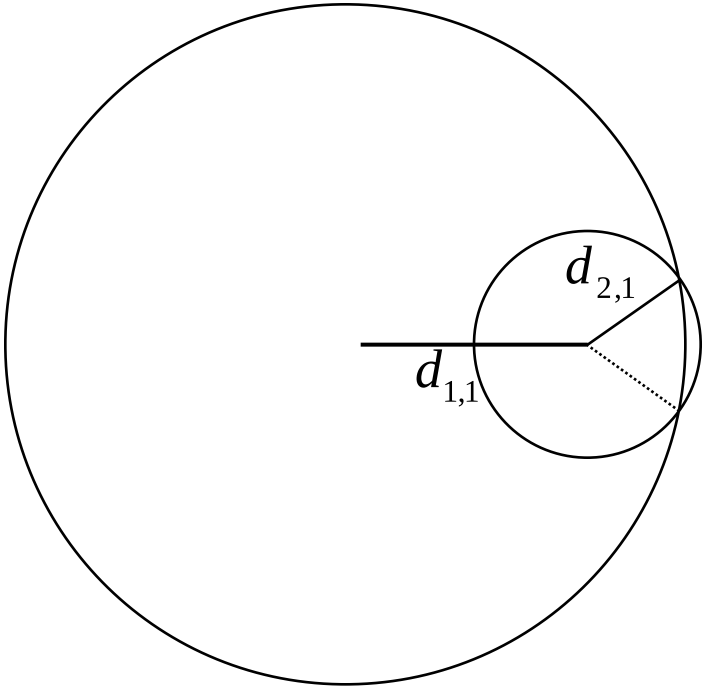
$$d_{i,1}, d_{i,2}, d_{i,3}$$

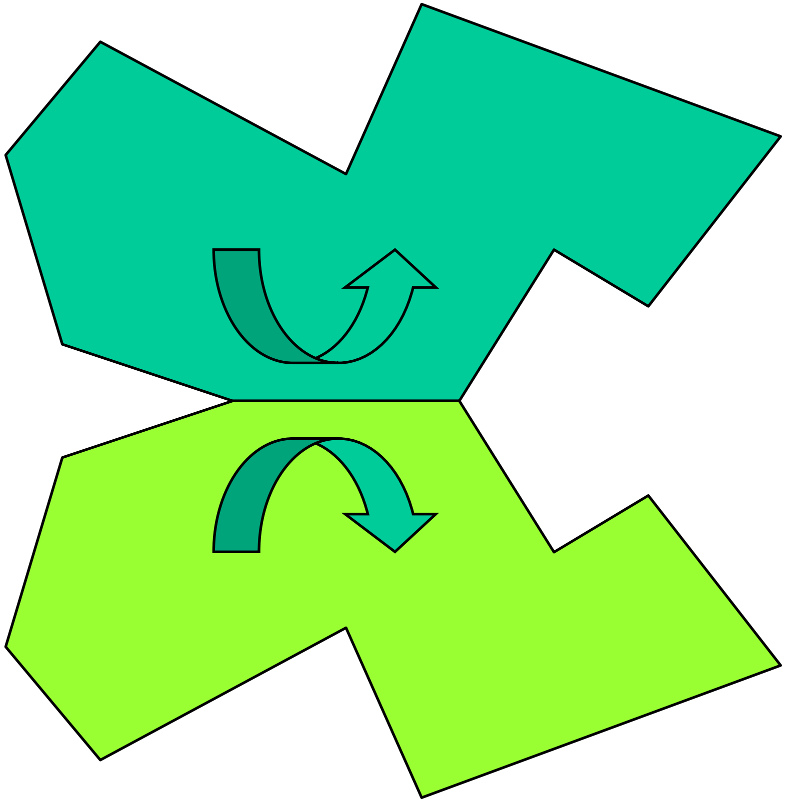
Because there is a unique intersection point of three circles.

This means, that a whole polygon can be uniquely generated by this basis elements iteratively.

With the first two distances we get two initial configurations. Then all further vertices will be unique.

The two initial configurations give two possible polygons, where one is just the mirror image of the other along the first edge as its axis.





As discrete functions of local support we derive *monomials* from distances between neighbouring vertices and hence we get invariants by summing these discrete functions of local support (DFLS) over all vertices:

$$\tilde{x}_{n_1,n_2,n_3,n_4} = \sum_{i\in\mathbb{V}} h(\Delta,\mathbf{x}_i) = \sum_{i\in\mathbb{V}} d_{i,1}^{n_1} d_{i,2}^{n_2} d_{i,3}^{n_3} d_{i,4}^{n_4}$$

Chosing the following 8 values for the exponents we would end up with a corresponding invariant feature vector and a set of 8 invariants:

i	n_1	n_2	n_3
$\tilde{x}_0$	1	0	0
$\tilde{x}_1$	1	1	0
$\tilde{x}_2$	1	0	1
$\tilde{x}_3$	1	1	1
$\tilde{x}_4$	2	0	0
$\tilde{x}_5$	2	1	0
$\tilde{x}_6$	2	0	1
$\tilde{x}_7$	2	1	1

We clearly recognize e.g. $\tilde{x}_0$

as the circumference of the polygon as an invariant.
For the above example of the letter F we get the following invariants:

$$\tilde{\mathbf{x}} = \begin{pmatrix} 21 & 44 & 83.82 & 68.6 & 184.6 & 665.3 & 149.5 & 751.6 \end{pmatrix}^T$$

How complete is this set of invariants?

Discrimination Performance, question of completeness

We expect a more and more complete feature space by summing over an increasing number of monomials of this basis elements

Looking at a triangle as the most simplest polygon one can show that the following three features derived from the three sides $\{a,b,c\}$ form a complete set of invariants:

$$\tilde{x}_0 = a + b + c, \quad \tilde{x}_1 = a^2 + b^2 + c^2, \quad \tilde{x}_2 = a^3 + b^3 + c^3$$

oder auch:

$$\tilde{x}_0 = a + b + c, \quad \tilde{x}_1 = ab + bc + ca, \quad \tilde{x}_2 = abc$$

The last features are equivalent to the elementary symmetrical polynomials in 3 variables which are a complete set of invariants with respect to all permutations.

Elementarsymmetrisches Polynom

Ein Polynom kann eindeutig durch seine Wurzeln oder auch durch seine Koeffizienten definiert werden (vollständige Invarianten):

$$(x-a)(x-b)(x-c) = x^3 \underbrace{-(a+b+c)}_{p_2} x^2 + \underbrace{(ab+bc+ca)}_{p_1} x \underbrace{-abc}_{p_0}$$

Das Polynom (und damit auch seine Koeffizienten) ist invariant gegenüber einer beliebigen Permutation der Wurzeln!
Somit ergibt sich folgende Korrespondenz:

Alle Permutationen der 3 Wurzeln a,b,c (symmetr. Gruppe)	○—●	Koeffizienten des elementarsymmetrischen Polynoms: $p_0 = -abc$, $p_1 = ab+bc+ca$, $p_2 = -(a+b+c)$

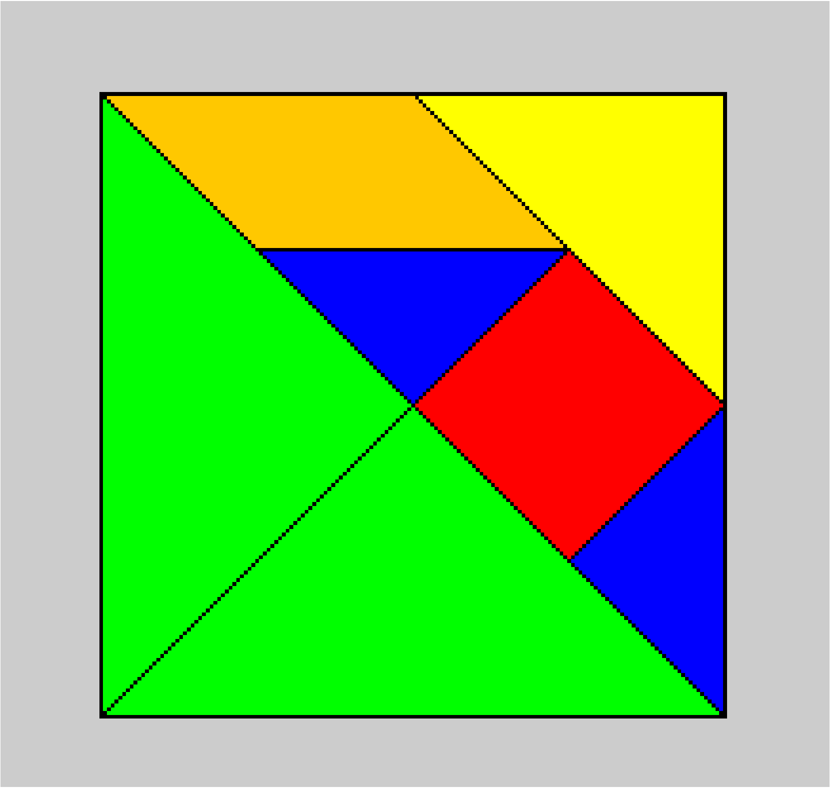
Completeness for finite Groups

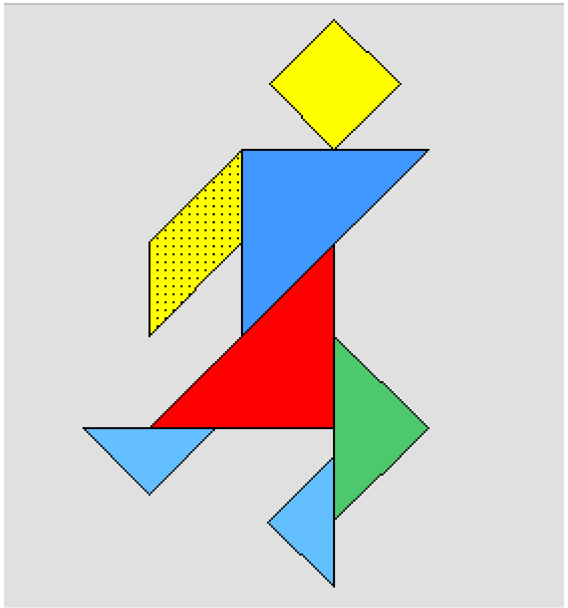
(Emmy Noether, 1916)

For finite Groups G with $|G|$ elements and patterns of dimensionality N the group averages over all monomials of degree $\leq |G|$ are complete and form a basis of the pattern space. The number of monomials is given by

$$\binom{N+|G|}{N}$$

Experiment: Object classification in a Tangram database

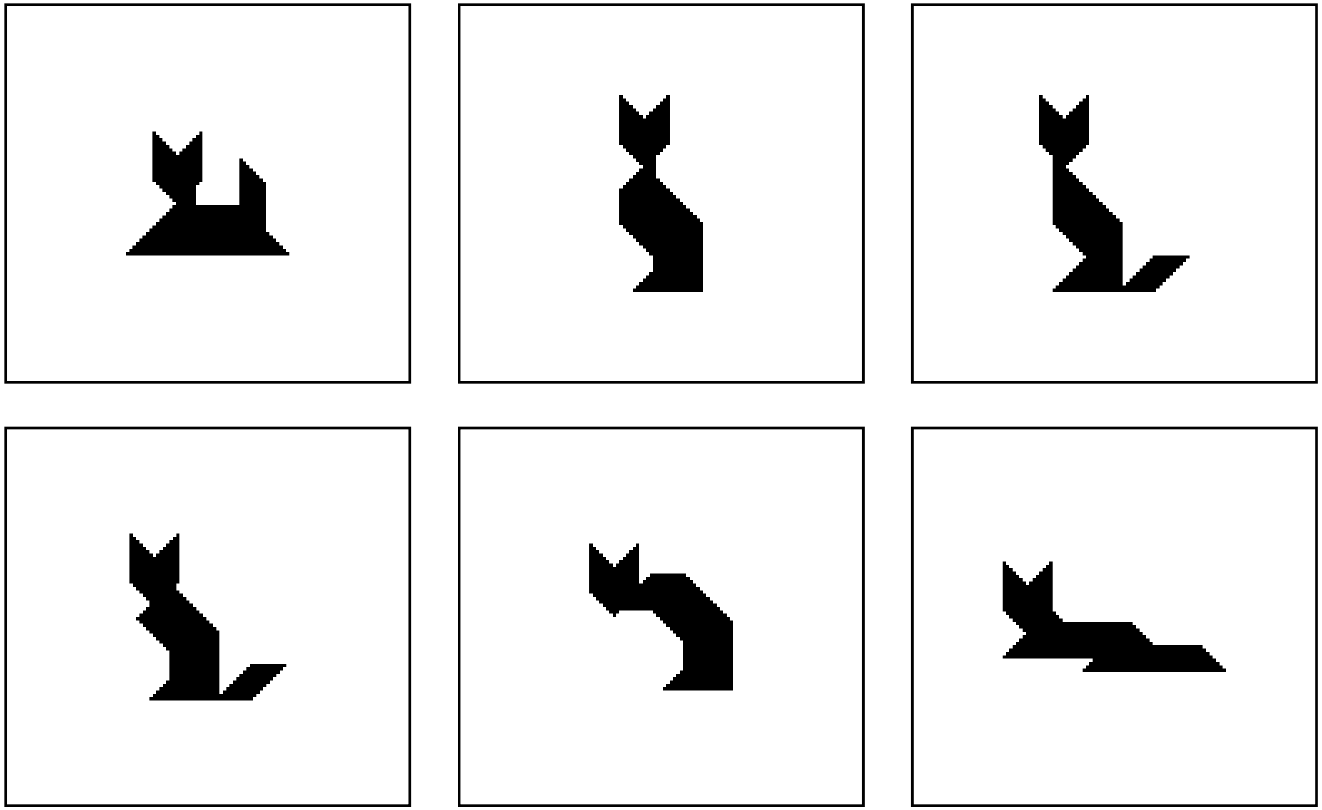


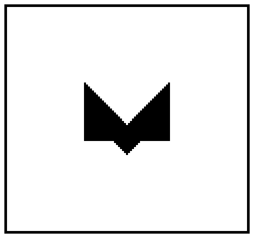


Take the outer contour as feature

Objects can not easily be discriminated with trivial geometric features!

Cats





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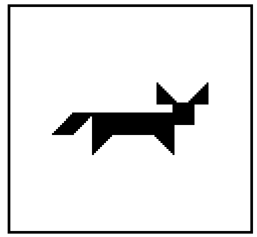

cow.png
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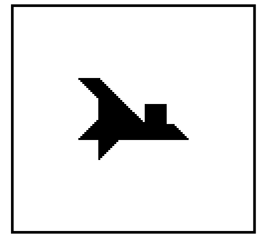

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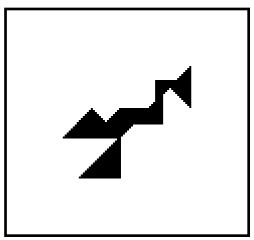

crab1.png

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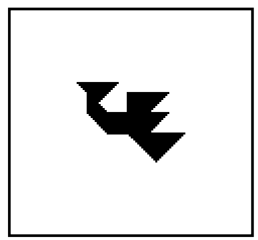

crab2.png

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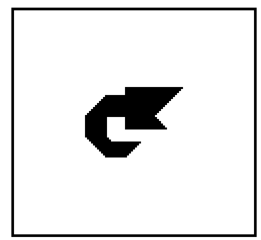

crab3.png

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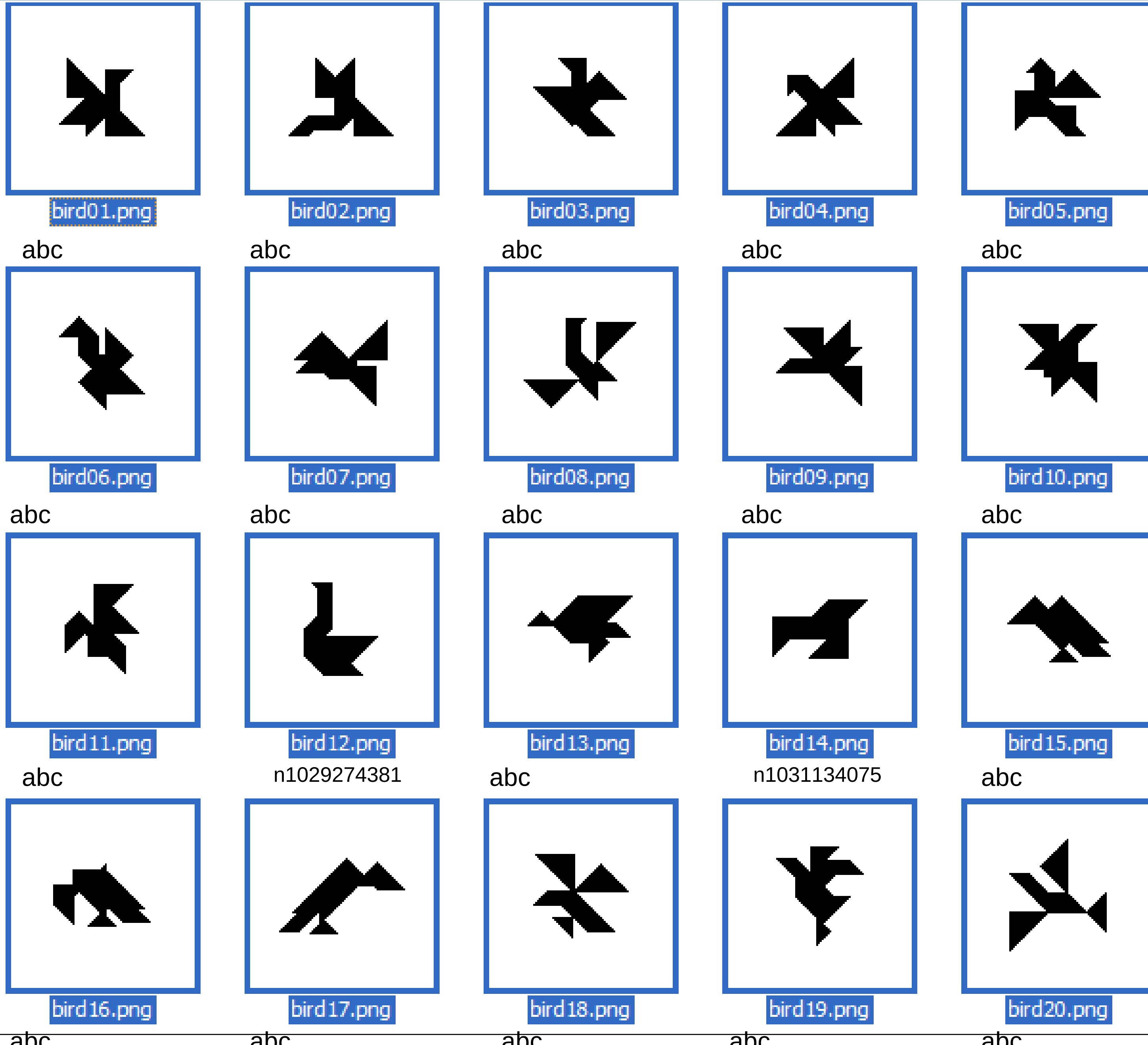
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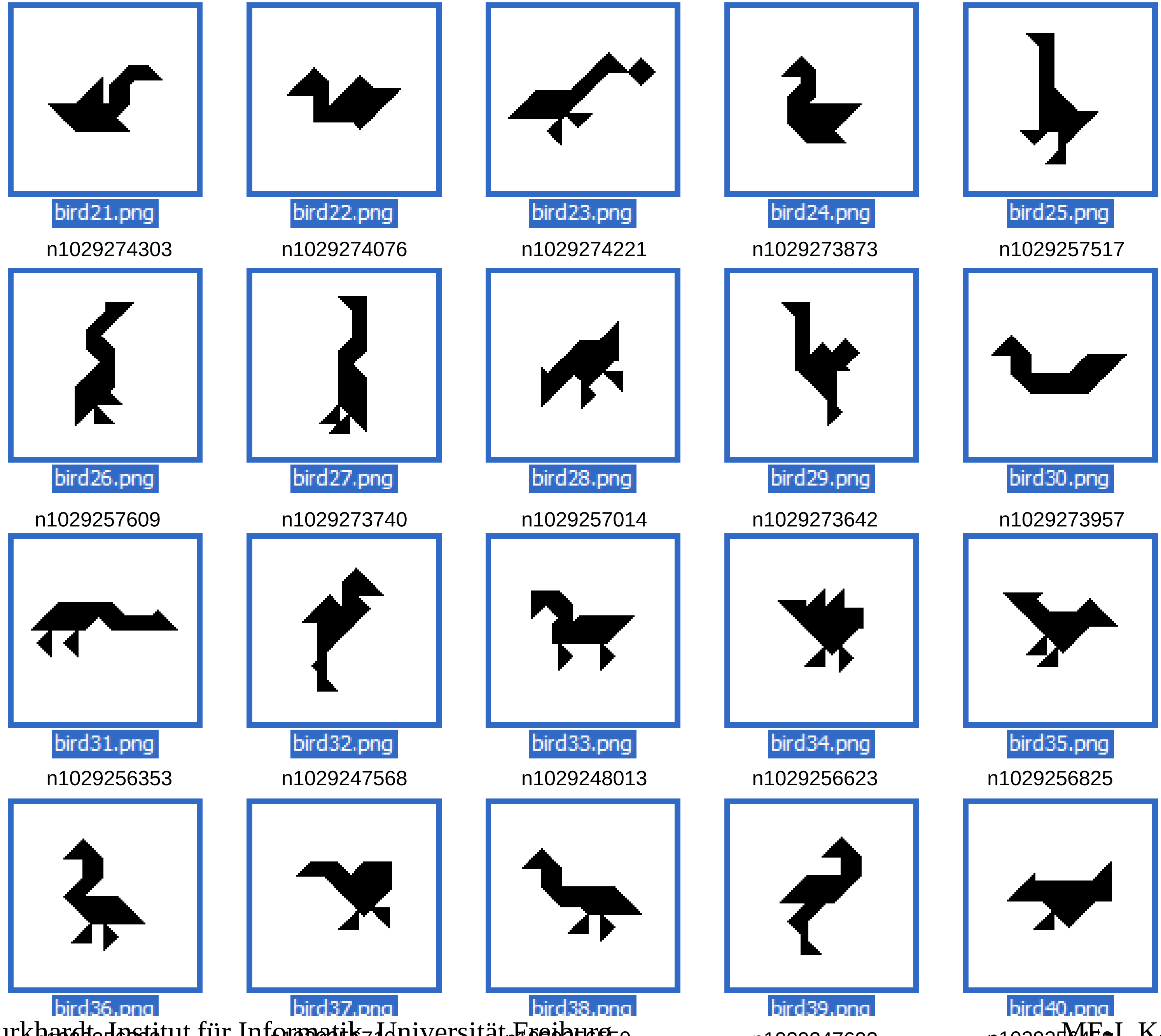
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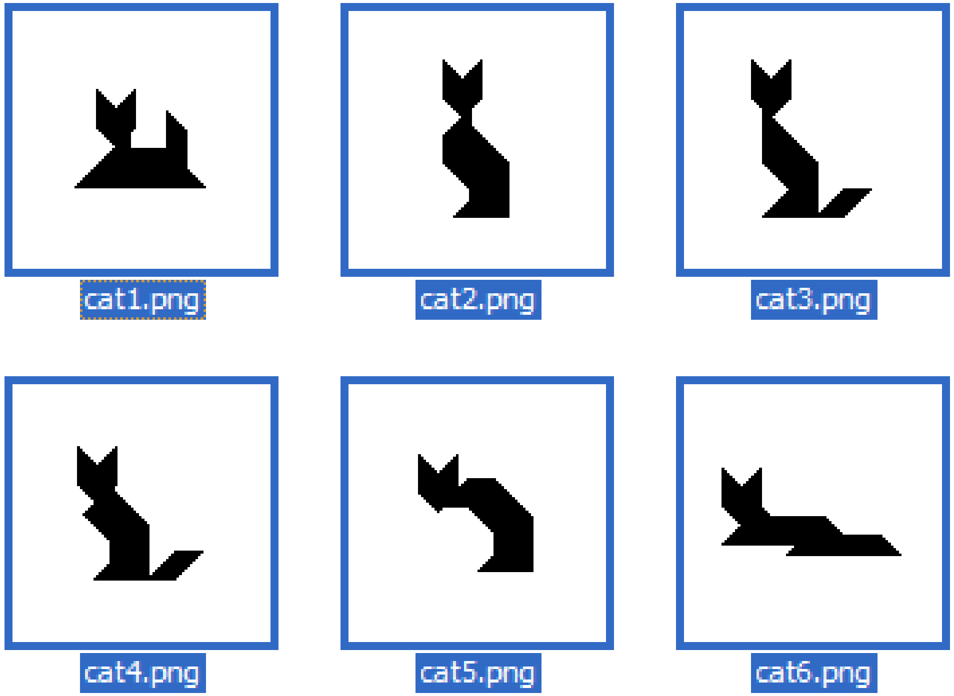

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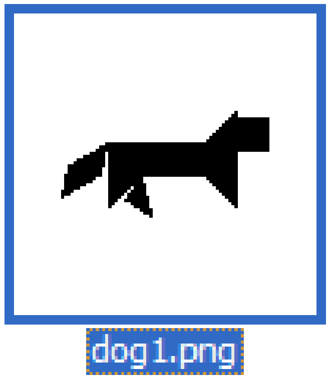

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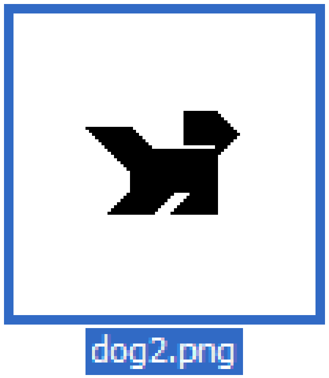

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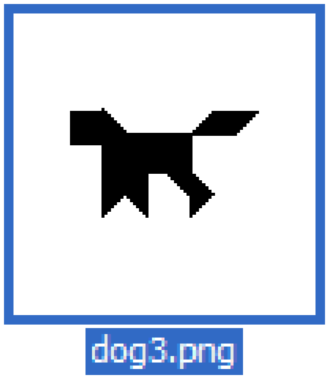

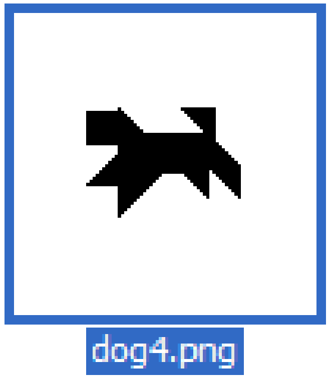

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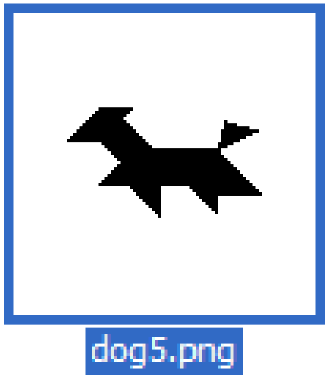

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fish02.png
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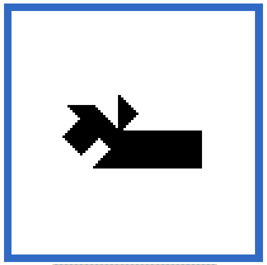
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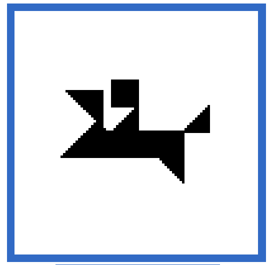
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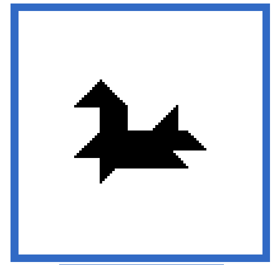
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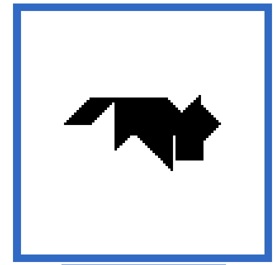
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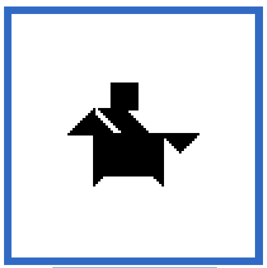
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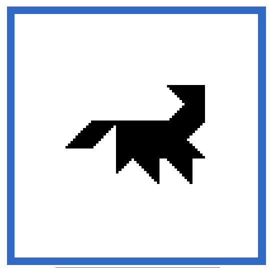
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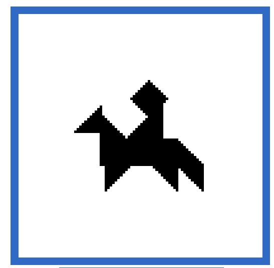
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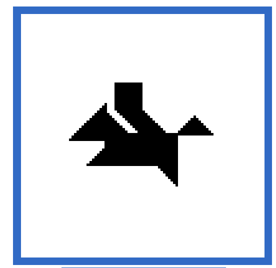
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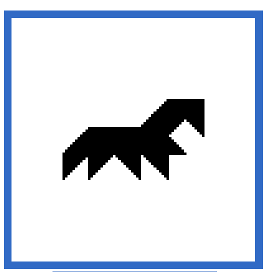
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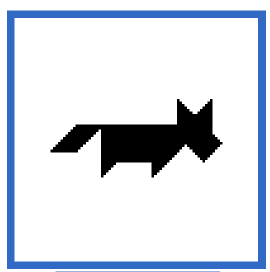
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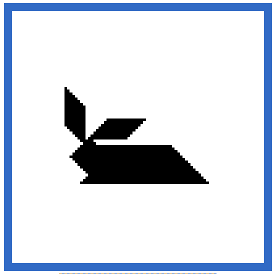
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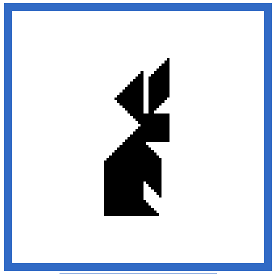
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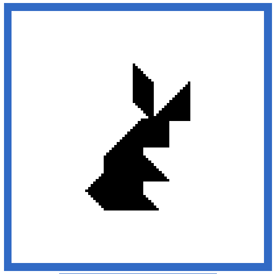
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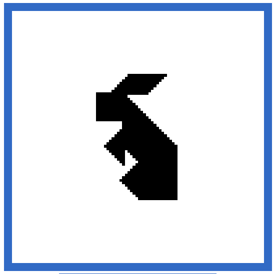
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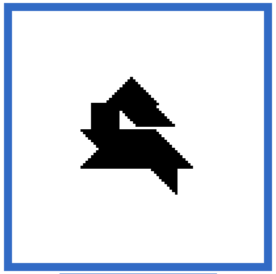
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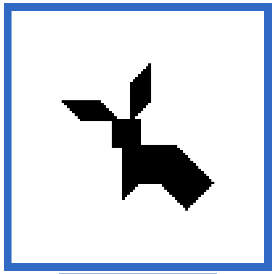
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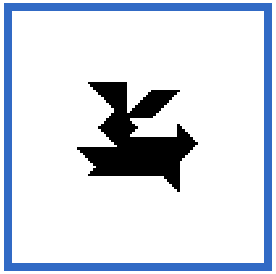
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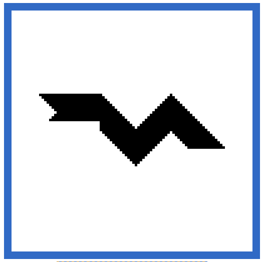

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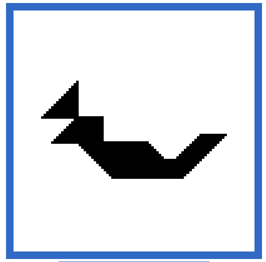
n1029696405



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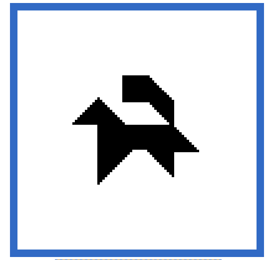

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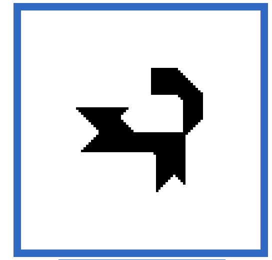

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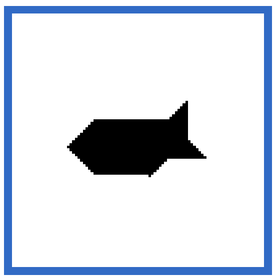

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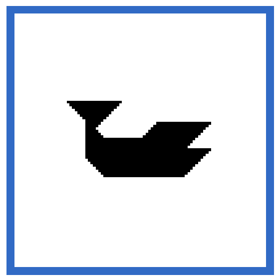

whale2.png

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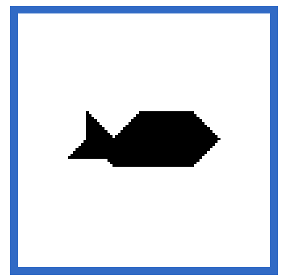

whale3.png

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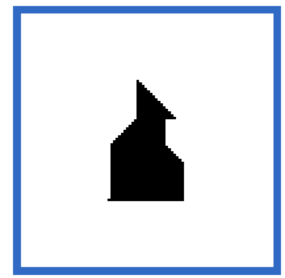

whale4.png

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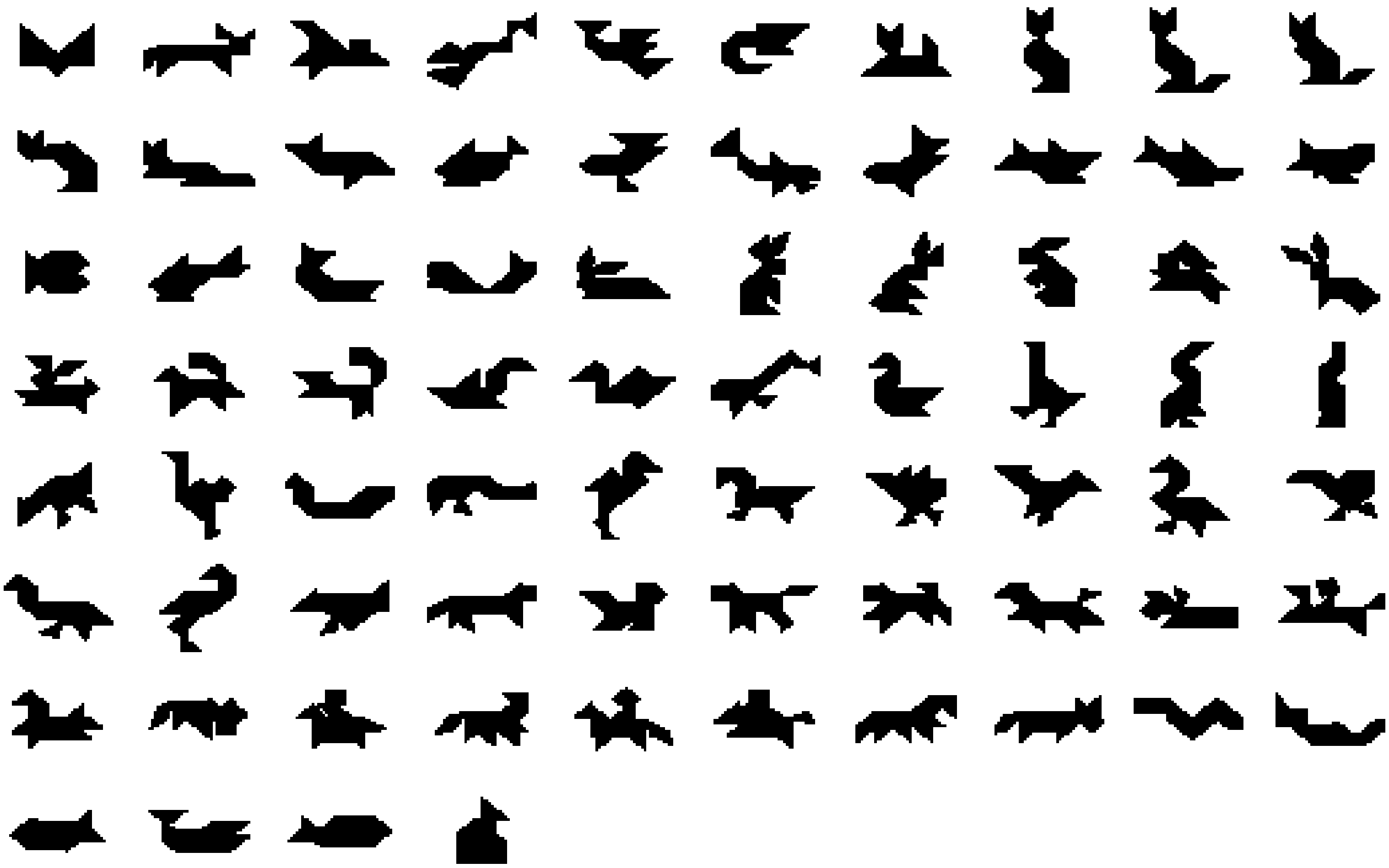


Fig. 6. The 74 tangrams used in the experiment.

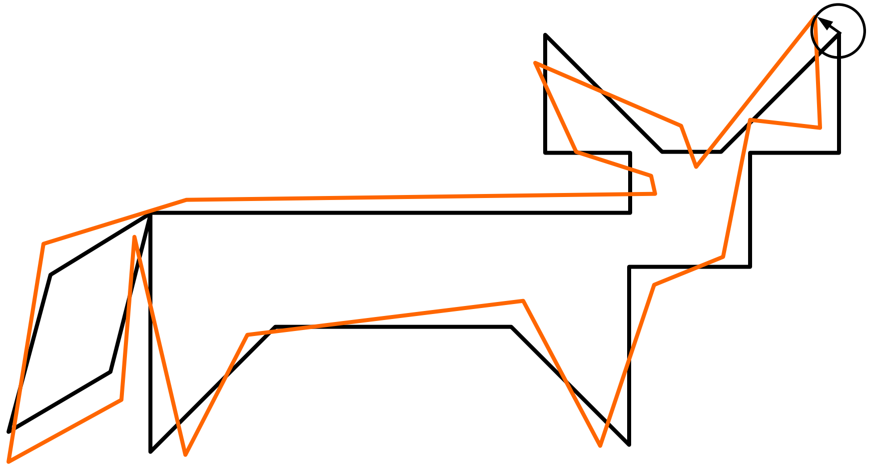
The experiments were conducted with three sets of 6, 10 and 14 invariants respectively using the following exponent table:

i	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	$\tilde{x}_8$	$\tilde{x}_9$	$\tilde{x}_{10}$	$\tilde{x}_{11}$	$\tilde{x}_{12}$	$\tilde{x}_{13}$
n_1	1	0	0	1	1	0	0	1	0	0	2	0	0	0
n_2	0	1	0	1	0	1	0	0	1	0	0	2	0	0
n_3	0	0	1	0	1	1	0	0	0	1	0	0	2	0
n_4	0	0	0	0	0	0	1	1	1	1	0	0	0	2

$$\tilde{x}_{n_1,n_2,n_3,n_4} = \sum_{i \in \mathbb{V}} h(\Delta, \mathbf{x}_i) = \sum_{i \in \mathbb{V}} d_{i,1}^{n_1} d_{i,2}^{n_2} d_{i,3}^{n_3} d_{i,4}^{n_4}$$

The classification performance was measured against additive noise of 5%, 10% and 20%.

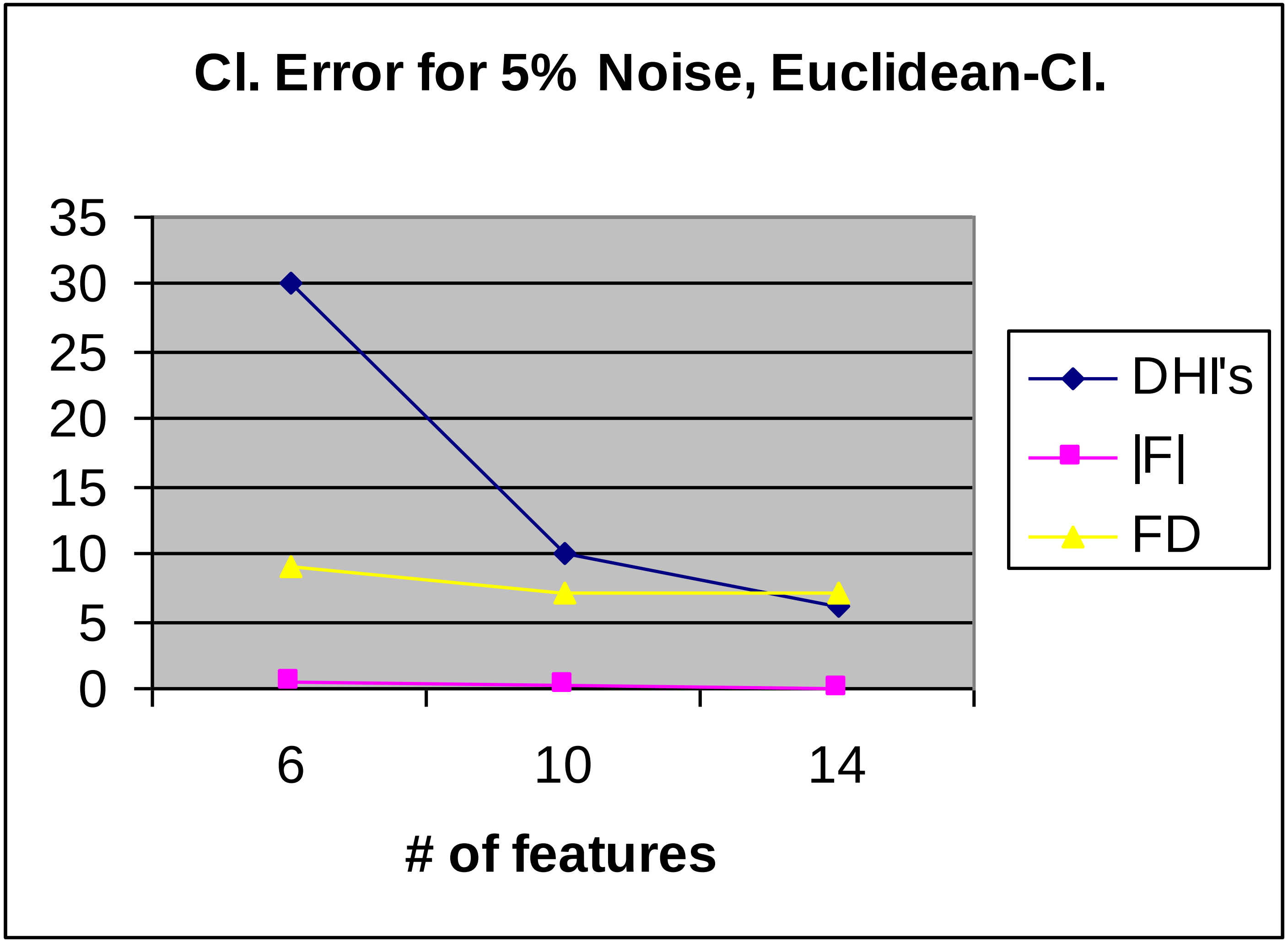
10% Noise

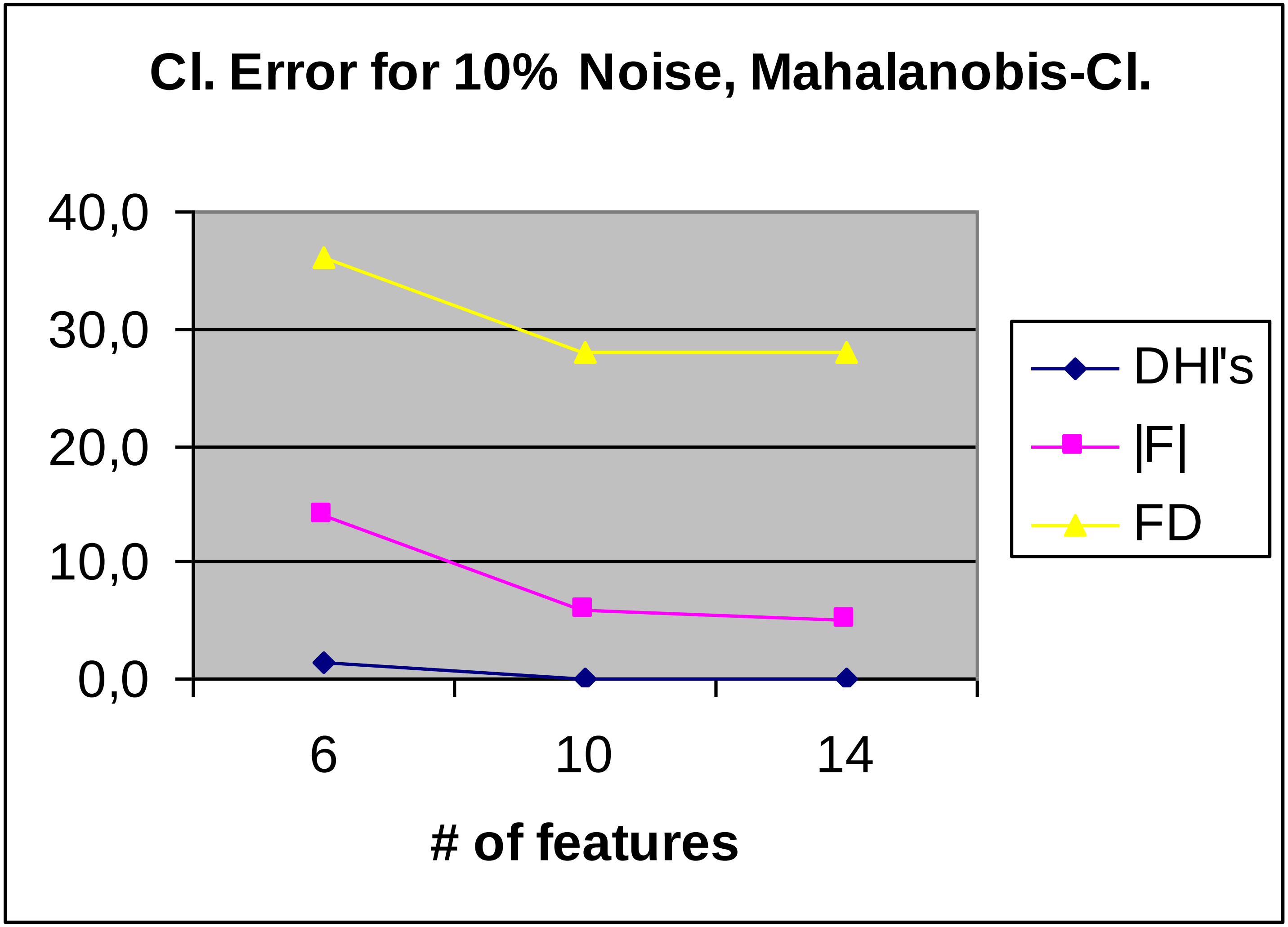


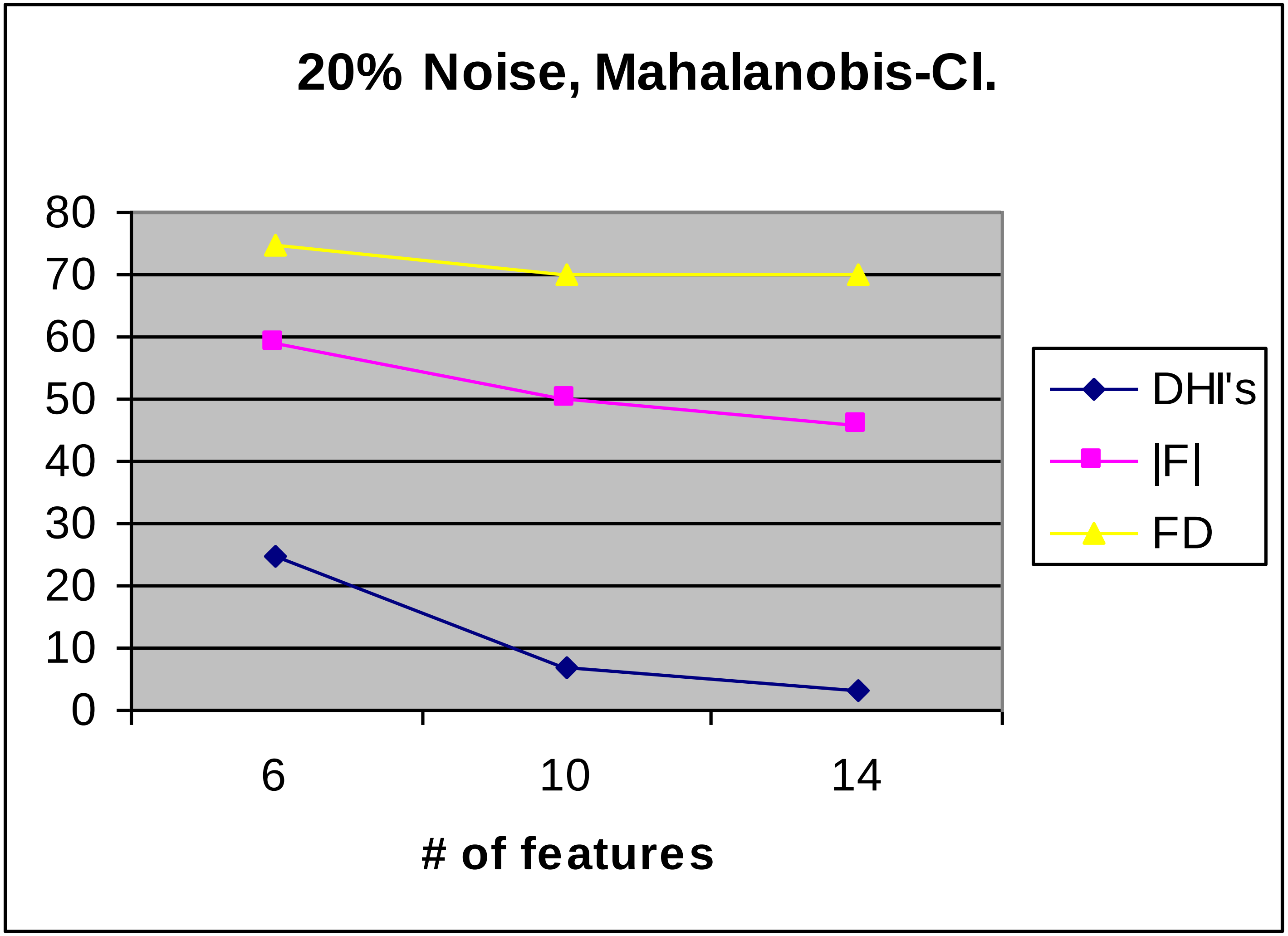
Classification error (in percent) for 5%, 10% and 20% noise for 74 tangrams with a Euclidean (E) and a Mahalanobis (M) Classifier

noise (in percent)	# of invariants	metric	DHI‘s class. error in %	FD error in %	|F| error in %
5	6	E	30	9	0.5
5	10	E	10	7	0.2
5	14	E	6	7	0
10	6	M	1.5	36	14
10	10	M	0	28	6
10	14	M	0	28	5
20	6	M	25	75	59
20	10	M	7	70	50
20	14	M	3	70	46

Empirical evaluation for the degree of completeness!

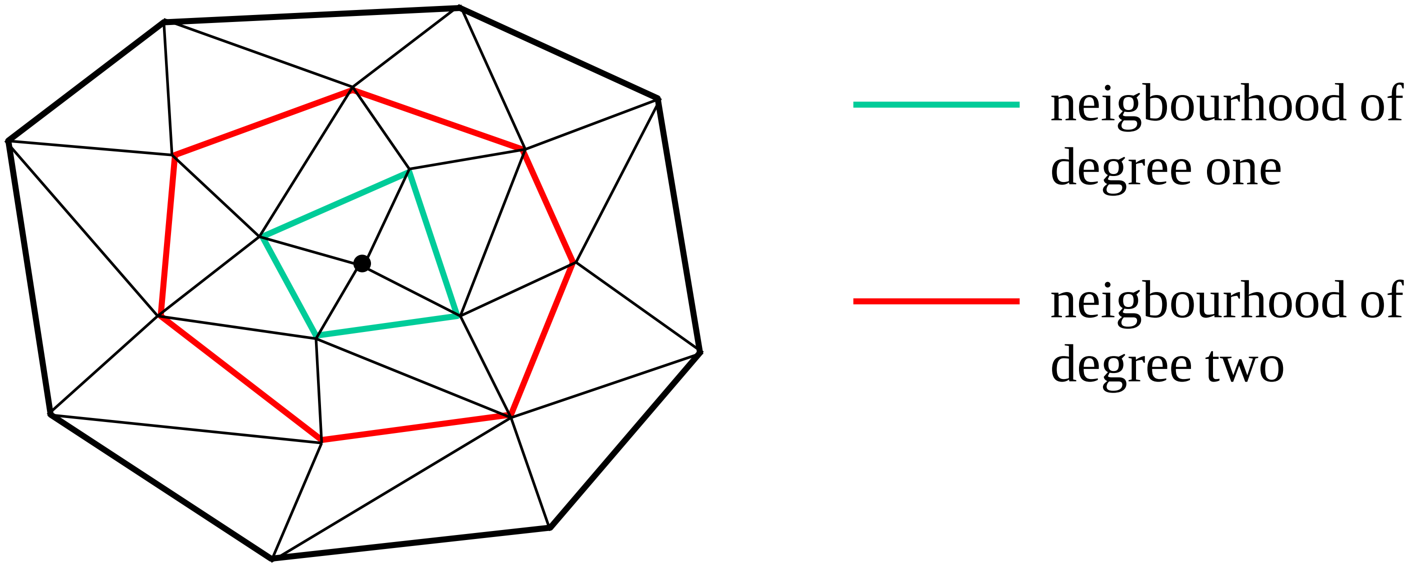

Cl. Error for 5% Noise, Euclidean-Cl.
35
30
25
20
15
10
5
0
6
10
14
# of features
DHI's
|F|
FD

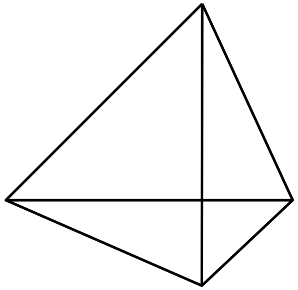

Cl. Error for 10% Noise, Mahalanobis-Cl.
40,0
30,0
20,0
10,0
0,0
6
10
14
# of features
DHI's
|F|
FD



20% Noise, Mahalanobis-Cl.
80
70
60
50
40
30
20
10
0
6
10
14
# of features
DHI's
|F|
FD

3D-Meshes

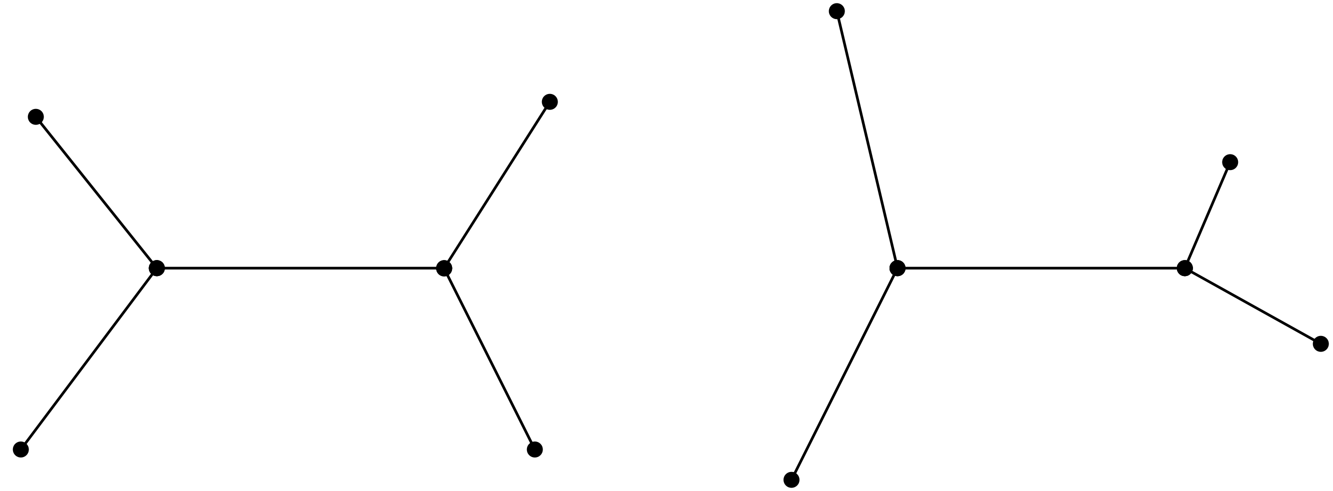




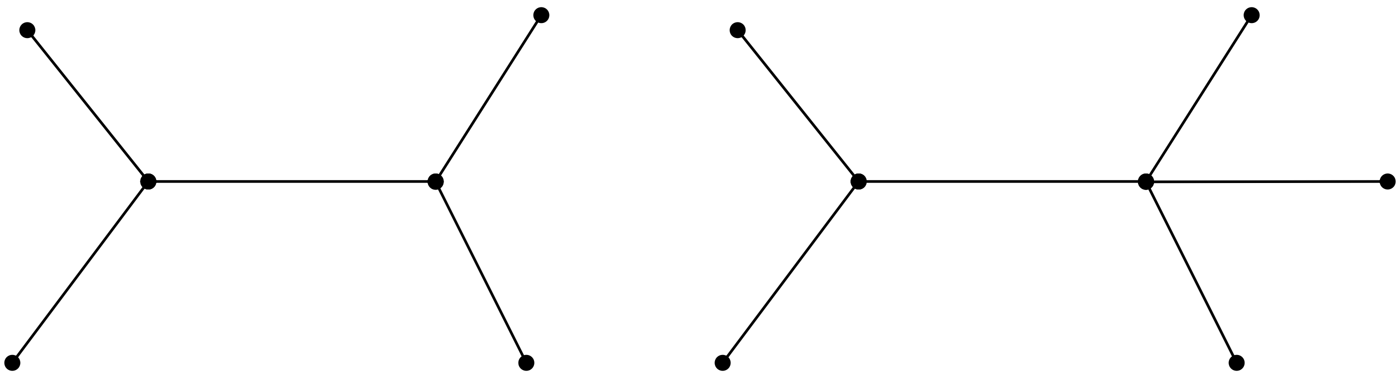
tetrahedron as the basic building block for a polyhedron

Topologically equivalent structures and (TE) non topologically equivalent structures (NTE)

TE



NTE



Properties

- If we constrain our calculation to a finite number of invariants we end up with a simple *linear complexity* in the number of vertices. This holds if the local neighborhood of vertices is resolved already by the given data structure; otherwise the cost for resolving local neighborhoods must be added.
- In contrast to *graph matching* algorithms we apply here algebraic techniques to solve the problem. This has the advantage that we can apply *hierarchical searches* for retrieval tasks, namely, to start only with one feature and hopefully eliminate already a large number of objects and then continue with an increasing number of features etc.

Conclusion

- In this paper we have introduced a novel set of invariants for discrete structures in 2D and 3D.
- The construction is a rigorous extension of Haar integrals over transformation groups to Dirac Delta Functions.
- The resulting invariants can easily be calculated with linear complexity in the number of vertices.
- The proposed approach has the potential to be extended to other discrete structures and even to the more general case of weighted graphs.

Literature: (http://lmb.informatik.uni-freiburg.de)

(1) S. Siggelkow and H. Burkhardt. Image retrieval based on local invariant features. In Proceedings of the IASTED International Conference on Signal and Image Processing (SIP) 1998, pages 369-373, Las Vegas, Nevada, USA, October 1998. IASTED.

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